

Equivariant weight filtration for real algebraic varieties with action

Fabien Priziac

Abstract

We show the existence of an equivariant weight filtration on the equivariant homology of real algebraic varieties equipped with a finite group action, by applying group homology onto the weight complex of McCrory and Parusiński. The group action enriches the information contained in the induced equivariant weight spectral sequence from which we can not extract finite additivity directly if the group is of even order.

Still, we manage to construct additive invariants from its elementary bricks, and prove that they are induced from the very equivariant geometry of real algebraic varieties with action, and that they coincide with Fichou's equivariant virtual Betti numbers in some cases. In the case of the two-elements group, we recover these through the use of globally invariant chains and the equivariant version of Guillén and Navarro Aznar's extension criterion.

1 Introduction

In [3], P. Deligne established the existence of a so-called weight filtration on the rational cohomology of complex algebraic varieties. A real analog of this filtration was introduced by B. Totaro on the Borel-Moore \mathbb{Z}_2 -homology of real algebraic varieties in [14]. Using an extension criterion for functors defined on nonsingular varieties by F. Guillén and V. Navarro Aznar ([6]), C. McCrory and A. Parusiński showed in [12] that this weight filtration for real algebraic varieties can be induced from a functorial (with respect to proper regular morphisms) filtered chain complex, defined up to filtered quasi-isomorphism. Considering the associated spectral sequence (that does not converge on order two, contrary to the complex frame), they highlight all the richness of the information contained in this invariant. In particular, McCrory and Parusiński extract from the real weight spectral sequence their additive virtual Betti numbers ([11]). Furthermore, they realize the weight complex from the chain level, using a so-called geometric filtration based on resolution of singularities ([7]), which coincides on the real algebraic varieties with a filtration defined on the category of \mathcal{AS} -sets ([8], [9]) and continuous proper maps with \mathcal{AS} -graph, using Nash-constructible functions ([10], [12]).

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In [13], we considered real algebraic varieties equipped with a finite group action. Thanks to the functoriality of the weight complex, we equipped it with the induced group action and, using an equivariant version of the extension criterion of Guillén and Navarro Aznar (a result that we recall as theorem 2.4), we showed the unicity of this weight complex with action with respect to extension, acyclicity and additivity properties, up to equivariant filtered quasi-isomorphism (theorem 2.3). Focusing on its realization through the Nash-constructible filtration, we obtained a filtered version of the Smith short exact sequence for an algebraic involution (theorem 2.6), relating isomorphically the filtered invariant chains and the filtered chains of the arc-symmetric quotient in the compact and fixed point free case (proposition 2.7).

In this paper, we apply some functor (definition and proposition 3.4) to the weight complex with action (definition 3.15) in order to induce a weight filtration on the equivariant homology of real algebraic varieties equipped with a finite group action. We show the unicity of the equivariant weight complex, with respect to extension, acyclicity and additivity properties, similarly to the previous frames (theorem 3.16). Though, taking a particular look at the induced equivariant weight spectral sequence (subsection 3.3), we understand that the action, embodied by the group homology of the considered group, enriches considerably the information contained in the equivariant weight invariants. In particular, some significative differences appear between the equivariant weight spectral sequence and the non-equivariant one. Indeed, the former is no longer left-bounded (corollaries 3.19 and 3.20). As a consequence, in the equivariant case, the long exact sequences of additivity are not finite in general and, even for compact nonsingular varieties, the spectral sequence does not degenerate on this page either (proposition 3.23). This prevents us to recover some additive invariants directly from the equivariant weight spectral sequence's page 2, unlike the non-equivariant frame.

In the final step of this work, we then propose means to grasp some additivity among elementary bricks of the equivariant weight spectral sequence, starting from the remark that its terms on page 2 are no more than the group homology of the considered finite group with coefficients in the weight spectral sequence (proposition 3.17). Taking the associated spectral sequences of group homology and considering the Nash-constructible filtration as the realization of the equivariant weight complex, finite long exact sequences of additivity appear on the rows of one of them (proposition 4.2). Combining these additivities in a certain way, we produce additive invariants B_k^G (definition 4.4). We show that each of them is the Euler characteristic of some other spectral sequence, constructed from the very equivariant geometry of real algebraic varieties with action (subsection 4.2 and theorem 4.9). The point here is that, by this approach, we manage to get some additive invariants of real algebraic varieties with action both obtained directly from their filtered semi-algebraic chains, and related to the equivariant weight spectral sequence. It should be stressed that the realization of the equivariant weight complex through the Nash-constructible filtration is crucial.

In the two-elements group case, computing the spectral sequences inducing the B_k^G 's by using the Smith Nash-constructible exact sequence, we get a formula that highlights the importance of the consideration of invariant chains (proposition 4.11). Furthermore, we use it to show that the invariants B_k^G coincide with G. Fichou's additive equivariant virtual Betti numbers in some cases (theorem 4.14). Would it be the case for all the equivariant virtual Betti numbers, and we would have a better geometric understanding of these additive invariants from the equivariant geometric structure at the filtered semi-algebraic chain level.

In the last point 4.4, we take an other point of view, starting from the invariant chains of a real algebraic variety equipped with an algebraic involution. Using the equivariant extension criterion 2.4 again, we show the existence of what we call an invariant weight complex (proposition 4.16), inducing a filtration of the homology of invariant chains. The induced spectral sequence is here bounded and we can extract additive invariants related to the equivariant virtual Betti numbers through the virtual Betti numbers of the fixed-points set (proposition 4.18 and theorem 4.19). Though, even if we manage to recover the equivariant virtual Betti numbers by this approach, it is non-effective in the sense that they come from a chain complex only defined up to filtered quasi-isomorphism, forcing us to consider acyclicity and additivity, that is resolutions of singularities and inclusions, to compute them, a way that is similar to what Fichou did in [5] to prove their existence. Consequently, future work will have to be done to obtain an realization at the chain level of the invariant weight complex. Whether the Nash-constructible filtration could be such a chain realization for the invariant weight complex seems furthermore to be the question that links this construction, the construction of additivity from the equivariant weight spectral sequence, and the equivariant virtual Betti numbers for the two-elements group. This confirms the central role that seems to be played by the Nash-constructible filtration.

We start this paper by recalling the principal definitions and properties from [13] about the weight complex with action that we shall need here. In particular, we give the equivariant extension criterion and the Smith Nash-constructible short exact sequence for an algebraic involution.

In section 3, after giving some words about group cohomology and homology, we define a functor denoted by L together with the equivariant homology and the associated spectral sequences. Showing the compatibility of L with filtered categories, we apply it to the weight complex with action to obtain an equivariant weight complex. We then focus on the induced equivariant weight spectral sequence, extracting the tools that we will use to identify additivities in the last part. We give also some examples for the computation of the equivariant weight filtration, and justify that the odd-order group case is the simple one.

Finally, in section 4, we explain how we can construct additive invariants from the elementary bricks of the equivariant weight spectral sequence, and recover them from spectral sequences induced by the equivariant geometry of real algebraic varieties with action, embodied by the Nash-constructible filtration (with action). After a computation, we show that in some cases, they equal the equivariant virtual Betti numbers. In the last subsection, we recover these for $G = \mathbb{Z}/2\mathbb{Z}$, considering the invariant chains.

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2 Weight complex with action

In this section, we recall the important results from [13], in particular the existence and unicity of the weight complex with action (2.3) and the exactness of what we called the Smith Nash-

constructible short sequence for an involution (2.6).

For the reader's convenience, we first recall the definition of the complex of semialgebraic chains with closed supports, before equipping it with a group action.

2.1 Semialgebraic chains with action

We recall the definition from [12], Appendix, of the semialgebraic chains with closed supports of a semialgebraic subset X of the set of real points of a real algebraic variety. Here, a real algebraic variety is a reduced separated scheme of finite type on \mathbb{R} .

Definition 2.1. *For all $k \geq 0$, we denote by $C_k(X)$ the quotient of the vector space on \mathbb{Z}_2 generated by the closed semialgebraic subsets of X of dimension $\leq k$, by the relations*

- *the sum $A + B$ is equivalent to the closure $cl_X(A \div B)$ in X (with respect to the strong topology) of the symmetric difference of A and B ,*
- *the class of A is zero if the dimension of A is strictly smaller than k .*

A semialgebraic chain with closed supports of dimension k is by definition an equivalence class of $C_k(X)$. Any chain c of $C_k(X)$ can be written as the class of a closed semialgebraic subset A of X of dimension $\leq k$, denoted by $[A]$.

The boundary operator $\partial_k : C_k(X) \rightarrow C_{k-1}(X)$ is defined by $\partial_k c = [\partial A]$, if $c = [A] \in C_k(X)$, where ∂A denotes the semialgebraic boundary $\{x \in A \mid \Lambda \mathbf{1}_A(X) \equiv 1 \pmod{2}\}$ of A (Λ is the link on the constructible functions, cf [10]).

Consider now a group G acting on X by semialgebraic homeomorphisms. By functoriality of the semialgebraic chains with closed supports, an action by linear isomorphisms is induced on the complex $C_*(X)$.

The complex $C_*(X)$ equipped with this action becomes a G -complex, that is the induced action of G on $C_*(X)$ commutes with its differential. Moreover, the semialgebraic chains with action are functorial with respect to equivariant proper continuous semialgebraic maps, and the operations of (equivariant) restriction, closure and pullback are equivariant.

For more details about semialgebraic chains, we refer to the appendix of [12], and for their equivariance, to the second section of [13].

2.2 Weight complex with action

Let G be a finite group.

We give the steps that led us to the construction and the unicity of the weight complex with action in [13], notably an equivariant version of the extension criterion [6] Théorème 2.2.2 of F. Guillén and V. Navarro Aznar (2.4).

The next definition is about the notations for the different equivariant categories we are working in.

Definition 2.2. *We denote by*

- $\mathbf{Sch}_c^G(\mathbb{R})$ the category of real algebraic varieties equipped with an action of G by algebraic isomorphisms -we call such objects real algebraic G -varieties- and equivariant regular proper morphisms,
- $\mathbf{Reg}_{comp}^G(\mathbb{R})$ the subcategory of compact nonsingular G -varieties,
- $\mathbf{V}^G(\mathbb{R})$ the subcategory of projective nonsingular G -varieties.

We denote also by

- \mathcal{C}^G the category of bounded G -complexes of \mathbb{Z}_2 -vector spaces equipped with an increasing bounded filtration by G -complexes with equivariant inclusions -we call such objects filtered G -complexes- and equivariant morphisms of filtered complexes,
- \mathcal{D}^G the category of bounded G -complexes and equivariant morphisms of complexes.

Here and from now, an action of G by algebraic isomorphisms on a real algebraic variety X will be an action by isomorphisms of schemes such that the orbit of any point in X is contained in an affine open subscheme.

In [13] Théorème 3.5, we defined a weight complex with action of G on the category of real algebraic G -varieties, by equipping McCrory-Parusiński's weight complex ([12] Theorem 1.1) with the action induced by functoriality :

Theorem 2.3. ([13] Théorème 3.5) *The functor*

$$F^{can}C_* : \mathbf{V}^G(\mathbb{R}) \longrightarrow H \circ \mathcal{C}^G ; X \mapsto F^{can}C_*(X)$$

admits an extension to a functor

$${}^G\mathcal{WC}_* : \mathbf{Sch}_c^G(\mathbb{R}) \longrightarrow H \circ \mathcal{C}^G$$

defined for all real algebraic G -varieties and all equivariant proper regular morphisms, which satisfies the following properties :

1. *Acyclicity : For any acyclic square*

$$\begin{array}{ccc} \tilde{Y} & \rightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \rightarrow & X \end{array} \quad (2.1)$$

in $\mathbf{Sch}_c^G(\mathbb{R})$, the simple filtered complex of the \square_1^+ -diagram in \mathcal{C}^G

$$\begin{array}{ccc} {}^G\mathcal{WC}_*(\tilde{Y}) & \rightarrow & {}^G\mathcal{WC}_*(\tilde{X}) \\ \downarrow & & \downarrow \\ {}^G\mathcal{WC}_*(Y) & \rightarrow & {}^G\mathcal{WC}_*(X) \end{array}$$

is acyclic (i.e. isomorphic to the zero complex in $H \circ \mathcal{C}^G$).

2. *Additivity* : For a equivariant closed inclusion $Y \subset X$, the simple filtered complex of the \square_0^+ -diagram in \mathcal{C}^G

$${}^G\mathcal{WC}_*(Y) \rightarrow {}^G\mathcal{WC}_*(X)$$

is isomorphic to ${}^G\mathcal{WC}_*(X \setminus Y)$.

Such a functor ${}^G\mathcal{WC}_*$ is unique up to a unique isomorphism of $H \circ \mathcal{C}^G$.

We denote by $H \circ \mathcal{C}^G$ the category \mathcal{C}^G localised with respect to equivariant filtered quasi-isomorphisms, that we will sometimes call quasi-isomorphisms of \mathcal{C}^G , i.e. the equivariant filtered morphisms between filtered G -complexes that induce an equivariant isomorphism at the level E^1 of the induced spectral sequences.

In order to show the unicity of the weight complex with action ${}^G\mathcal{WC}_*$, we used an equivariant version of the extension criterion [6] Théorème 2.2.2 of Guillén and Navarro Aznar (that McCrory and Parusiński used to show the unicity of their weight complex, cf [12] Theorem 1.1), which justifies the restriction to the case of a finite group for which there exists an equivariant compactification, a Chow-Hironaka lemma and an equivariant resolution of singularities in the category $\mathbf{Sch}_c^G(\mathbb{R})$ (by [4] Appendix) :

Theorem 2.4. ([13] Théorème 3.5.) *Let \mathcal{C} be a category of cohomological descent and*

$$F : \mathbf{V}^G(\mathbb{R}) \longrightarrow H \circ \mathcal{C}$$

be a contravariant Φ -rectified functor verifying

(F1) $F(\emptyset) = 0$, and the canonical morphism $F(X \sqcup Y) \rightarrow F(X) \times F(Y)$ is an isomorphism (in $H \circ \mathcal{C}$),

(F2) if X_\bullet is an elementary acyclic square of $\mathbf{V}^G(\mathbb{R})$, then $\mathbf{s}F(X_\bullet)$ is acyclic.

Then, there exists an extension of F to a contravariant Φ -rectified functor

$$F_c : \mathbf{Sch}_c^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}$$

such that :

1. if X_\bullet is an acyclic square of $\mathbf{Sch}_c^G(\mathbb{R})$, then $\mathbf{s}F_c(X_\bullet)$ is acyclic,
2. if Y is a closed subvariety of X stable under the action of G on X , we have a natural isomorphism (in $H \circ \mathcal{C}$)

$$F_c(X \setminus Y) \cong \mathbf{s}(F_c(X) \rightarrow F_c(Y)).$$

Furthermore, this extension is unique up to a unique isomorphism.

Remark 2.5. • For every real G -variety, we have an equivariant isomorphism $H_n({}^G\mathcal{WC}_*(X)) \cong H_n(X)$ for all $n \in \mathbb{Z}$.

- The weight filtration and spectral sequence are equipped with the action of G induced from the G -action on the weight complex.
- McCrory and Parusiński's geometric/Nash-constructible filtration on real algebraic varieties ([12] sections 2 and 3), equipped with the action of G induced by functoriality, realizes the weight complex with action.
- If X is a compact nonsingular G -variety, ${}^G\mathcal{WC}_*(X)$ is isomorphic to $F^{can}C_*(X)$ in $H \circ \mathcal{C}^G$, and it goes the same for all the realizations of the weight complex with action, in particular for the geometric/Nash-constructible filtration with action.

For more details about the weight complex with action, we refer to [13] section 3.

In the following, the weight complex with action will be simply denoted \mathcal{WC}_* if the context is explicit.

2.3 The Smith Nash-constructible exact sequence in the case $G = \mathbb{Z}/2\mathbb{Z}$

In the last point of this section, we recall results from [13] section 5, which claim that we can use the Nash-constructible filtration to implement a notion of regularity into the Smith short exact sequence of a real algebraic variety with an involution.

Precisely, let X be a real algebraic variety equipped with an algebraic involution σ , then we can split with some regularity any invariant chain into two parts exchanged by the action of $G := \{1, \sigma\} = \mathbb{Z}/2\mathbb{Z}$ (modulo the restriction to the fixed points set of X) :

Theorem 2.6. ([13] Théorème 5.5.) *Let $c \in (\mathcal{N}_\alpha C_k(X))^G$ be a chain of X of dimension k and of degree α with respect to the Nash-constructible filtration. Then there exists $c' \in \mathcal{N}_{\alpha+1}C_k(X)$ such that*

$$c = c|_{X^G} + (1 + \sigma)c'$$

(the restriction $c|_{X^G}$ is in $\mathcal{N}_\alpha C_k(X^G)$).

Consequently, for all α , the short sequence of complexes

$$0 \rightarrow \mathcal{N}_\alpha C_*(X^G) \oplus (1 + \sigma)T_*^{\alpha+1}(X) \rightarrow \mathcal{N}_\alpha C_*(X) \rightarrow (1 + \sigma)\mathcal{N}_\alpha C_*(X) \rightarrow 0,$$

where $T_k^{\alpha+1}(X) := \{c \in \mathcal{N}_{\alpha+1}C_k(X) \mid (1 + \sigma)c \in \mathcal{N}_\alpha C_k(X)\}$, is exact. We call this sequence the Smith Nash-constructible exact sequence of X of degree α .

The last property we recall is an interpretation of the Smith Nash-constructible exact sequence when the variety X is compact and the action of G is free. In this case, the quotient of the real points of X by the action of G (that we denote, by an abuse of notation, by X/G) is an arc-symmetric set and the invariant chains of X correspond to the chains of the quotient :

Proposition 2.7. ([13] Proposition 5.6.) *Let X be a compact real algebraic variety equipped with a fixed-point free action of $G = \mathbb{Z}/2\mathbb{Z}$. Then the morphism of filtered complexes*

$$(\mathcal{NC}_*(X))^G \rightarrow \mathcal{NC}_*(X/G),$$

induced by the quotient map $\pi : X \rightarrow X/G$, is a filtered isomorphism.

3 Equivariant weight filtration for G -real algebraic varieties

Let G be a finite group.

In this section, we construct a so-called weight filtration on the equivariant homology of real algebraic G -varieties defined by J. van Hamel in [15], applying on the weight complex with action (2.3) some functor which computes this equivariant homology, when applied to the complex of semialgebraic chains (3.4, 3.11).

Significative differences will appear between the equivariant weight complex and the non-equivariant one. In particular, unlike McCrory and Parusiński's weight spectral sequence ([12] section 1.3), the equivariant weight spectral sequence may not be bounded (3.24) and may not degenerate at level E^2 , even in the compact nonsingular case (3.23, 3.13).

Nevertheless, tools from equivariant homology, useful to have a better grasp of this invariant, will provide us new spectral sequences (trivial in the non-equivariant case) to study more deeply all the information we get in this equivariant frame (3.17).

3.1 Group (co)homology and the functor L

In a first subsection, we recall the basic background about group (co)homology with coefficients in a module and a chain complex that we will use to define the equivariant homology on a real algebraic variety with action.

We remind that if M is a $\mathbb{Z}[G]$ -module, the n^{th} group of cohomology of the group G with coefficients in M is given by

$$H^n(G, M) := \text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M) = H^n(\text{Hom}_{\mathbb{Z}[G]}(F_*, M)),$$

where F_* is a projective resolution of \mathbb{Z} by $\mathbb{Z}[G]$ -modules.

Remark 3.1. • For $n < 0$, $H^n(G, M) = 0$.

- If k is a ring and if M is a $k[G]$ -module, then for all $n \in \mathbb{Z}$,

$$H^n(G, M) = \text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M) \cong \text{Ext}_{k[G]}^n(k, M),$$

as an isomorphism of k -modules. In the rest of this paper, we will be considering \mathbb{Z}_2 -vector spaces equipped with a linear action of G , so there will be no difference whether we should consider a projective resolution of \mathbb{Z} by $\mathbb{Z}[G]$ -modules or a projective resolution of \mathbb{Z}_2 by $\mathbb{Z}_2[G]$ -modules

Example 3.2. ([1]) Let G a finite cyclic group of order d generated by σ . Then, if we denote $N := \sum_{1 \leq i \leq d} \sigma^i$, a projective resolution of \mathbb{Z} by $\mathbb{Z}[G]$ -modules is given by

$$\cdots \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z} \rightarrow 0,$$

and the cohomology of G with coefficients in a module M by

$$H^n(G, M) = \begin{cases} \frac{M^G}{NM} & \text{if } n \text{ is an even positive integer,} \\ \frac{\ker(M \xrightarrow{N} M)}{(\sigma-1)M} & \text{if } n \text{ is an odd positive integer,} \\ M^G & \text{if } n = 0. \end{cases}$$

Example 3.3. If $G = \mathbb{Z}/2\mathbb{Z}$ and if M is a $\mathbb{Z}_2[G]$ -module, the cohomology of G with coefficients in M is

$$H^n(G, M) = \begin{cases} M^G & \text{si } n = 0, \\ \frac{M^G}{(1+\sigma)M} & \text{si } n > 0. \end{cases}$$

For more details and background about group cohomology, see for instance [1] or [2].

Now we define the homology of G with coefficients in a G -chain complex of \mathbb{Z}_2 -vector spaces, using a functorial operation denoted by L . We denote by \mathcal{D}_- the category of bounded above chain complexes of \mathbb{Z}_2 -vector spaces, and $H \circ \mathcal{D}_-$ the category \mathcal{D}_- localized with respect to quasi-isomorphisms.

Definition and Proposition 3.4. Let K_* be in \mathcal{D}^G . Let $\dots \rightarrow F_2 \xrightarrow{\Delta_2} F_1 \xrightarrow{\Delta_1} F_0 \rightarrow \mathbb{Z} \rightarrow 0$ be a resolution of \mathbb{Z} by projective $\mathbb{Z}[G]$ -modules.

Then, the complex $L_*(K_*)$ is the total complex associated to the double complex

$$(Hom_G(F_{-p}, C_q))_{p,q \in \mathbb{Z}}$$

The operation $L : \mathcal{D}^G \rightarrow \mathcal{D}_-$; $K_* \mapsto L_*(K_*)$ is functorial.

Remark 3.5. For $G = \{e\}$, considering $\dots \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{id} \mathbb{Z} \rightarrow 0$ as a projective resolution, we obtain $L_*(K_*) = K_*$.

The homology of the group G with coefficients in a G -complex K_* is then defined as the homology of the complex $L_*(K_*)$ and denoted by $H_*(G, K_*)$.

The two spectral sequences associated to the double complex $(Hom(F_{-p}, C_q))_{p,q \in \mathbb{Z}}$ converge to this homology :

$$\left. \begin{aligned} {}_I E_{p,q}^2 &= H^{-p}(G, H_q(K_*)) \\ {}_{II} E_{p,q}^1 &= H^{-p}(G, K_q) \end{aligned} \right\} \implies H_{p+q}(G, K_*)$$

The first spectral sequence is the so-called Hochschild-Serre spectral sequence associated to the group G and the G -complex K_* . In particular, we can see here that, as the group cohomology with coefficients in a module does not depend on the considered projective resolution (of \mathbb{Z} by $\mathbb{Z}[G]$ -modules or of \mathbb{Z}_2 by $\mathbb{Z}_2[G]$ -modules), neither do the group homology with coefficients in a complex of \mathcal{D}^G and the functor $L : \mathcal{D}^G \rightarrow H \circ \mathcal{D}_-$; $K_* \mapsto L_*(K_*)$ (for convenience, we use the same notation L^G).

Hochschild-Serre spectral sequence allows us to prove that L preserves quasi-isomorphisms as well :

Proposition 3.6. An equivariant quasi-isomorphism $f : K_* \rightarrow M_*$ induces an isomorphism $H_*(G, K_*) \rightarrow H_*(G, M_*)$.

Proof. The equivariant quasi-isomorphism f induces an isomorphism from the level ${}_I E^2$ of the induced Hochschild-Serre spectral sequences, which converge to the homologies of G with coefficients in K_* and M_* respectively. \square

We denote again by L the induced functor $H \circ \mathcal{D}^G \rightarrow H \circ \mathcal{D}_-$.

Furthermore, we show that if we apply the functor L to a complex of \mathcal{C}^G , we can obtain a filtered complex (bounded above), and that this operation preserves filtered quasi-isomorphisms.

Define the category \mathcal{C}_- of bounded above complexes of \mathbb{Z}_2 -vector spaces equipped with an increasing bounded filtration, and morphisms of filtered complexes.

Proposition 3.7. *Let (K_*, J) be in \mathcal{C}^G . The equivariant increasing bounded filtration J of the G -complex K_* induces an increasing bounded filtration \mathcal{J} on the complex $L_*(K_*)$, defined by*

$$\mathcal{J}_\alpha L_k(K_*) := L_k(J_\alpha K_*).$$

In $H \circ \mathcal{C}_-$, the couple $(L_(K_*), \mathcal{J})$ is independent from the chosen projective resolution.*

Proof. We show that two projective resolutions give quasi-isomorphic filtered complexes in \mathcal{C}_- .

Let $(F_i)_i$ and $(F'_j)_j$ be two resolutions of \mathbb{Z} or \mathbb{Z}_2 by projective $\mathbb{Z}[G]$ -modules, respectively $\mathbb{Z}_2[G]$ -modules. We denote by $\mathcal{J}L_*(K_*)$ and $\mathcal{J}'L'_*(K_*)$ the respective associated filtered complexes, and by E^r and E'^r the respective induced spectral sequences.

We have

$$E_{p,q}^0 = \frac{\mathcal{J}_p L_{p+q}}{\mathcal{J}_{p-1} L_{p+q}} = \frac{\bigoplus_{a+b=p+q} \text{Hom}_G(F_{-a}, J_p K_b)}{\bigoplus_{a+b=p+q} \text{Hom}_G(F_{-a}, J_{p-1} K_b)} = \bigoplus_{a+b=p+q} \text{Hom}_G\left(F_{-a}, \frac{J_p K_b}{J_{p-1} K_b}\right)$$

because the G -modules F_i are projective. Then we have $E_{p,*}^0 = L_{p+*}\left(\frac{J_p K_*}{J_{p-1} K_*}\right)$ and $E_{p,*}'^0 = L_{p+*}'\left(\frac{J_p K_*}{J_{p-1} K_*}\right)$ for all $p \in \mathbb{Z}$. The homologies of these two complexes give the homology of the group G with coefficients in the complex $\frac{J_p K_*}{J_{p-1} K_*}$, inducing an isomorphism between E^1 et E'^1 . \square

We use this computation to show that the functor L preserves filtered quasi-isomorphisms :

Proposition 3.8. *The operation L induces a functor*

$$H \circ \mathcal{C}^G \rightarrow H \circ \mathcal{C}_- ; (K_*, J) \mapsto (L_*(K_*), \mathcal{J})$$

that we denote again by L .

Proof. Let $\varphi : K_* \rightarrow M_*$ be a quasi-isomorphism in \mathcal{C}^G and J and I be the respective filtrations of K_* and M_* . We just computed

$$E_{p,q}^0(L_*(K_*)) = L_{p+q}\left(E_{p,*-p}^0(K_*)\right) \text{ and } E_{p,q}^0(L_*(M_*)) = L_{p+q}\left(E_{p,*-p}^0(M_*)\right).$$

But, for all $p \in \mathbb{Z}$, the morphism $\varphi : K_* \rightarrow M_*$ induces an equivariant quasi-isomorphism $E_{p,*-p}^0(K_*) \rightarrow E_{p,*-p}^0(M_*)$, and then we use the fact that L preserves quasi-isomorphisms to conclude. \square

We apply L to the canonical filtration. We notice that the induced spectral sequence coincides with Hochschild-Serre spectral sequence :

Lemma 3.9. *Let (K_*, ∂_*) be a G -complex equipped with the canonical filtration F^{can} . We consider the induced filtered complex $\mathcal{F}^{can} L_*(K_*)$, and denote by E the induced spectral sequence. We have, for all $p, q \in \mathbb{Z}$,*

$$E_{p,q}^1 = H^{-2p-q}(G, H_{-p}(K_*)),$$

and, for all $r \geq 1$,

$$E_{p,q}^r = {}_tE_{2p+q, -p}^{r+1}.$$

Proof. Let (F_*, Δ_*) be a projective resolution on $\mathbb{Z}[G]$ of \mathbb{Z} (or on $\mathbb{Z}_2[G]$ of \mathbb{Z}_2). Let p in \mathbb{Z} .

By a direct computation and using the fact that the F_i 's are projective, we find that the complex $E_{p, *-p}^0$ is the mapping cone of the morphism

$$\phi : \text{Hom}_G(F_{-(p+*)}, K_{-p+1}/\ker \partial_{-p+1}) \rightarrow \text{Hom}_G(F_{-(p+*)}, \ker \partial_{-p}).$$

On the other hand, again because the F_i 's are projective, the short sequence of complexes

$$0 \rightarrow \text{Hom}_G(F_{-(p+*)}, K_{-p+1}/\ker \partial_{-p+1}) \rightarrow \text{Hom}_G(F_{-(p+*)}, \ker \partial_{-p}) \rightarrow \text{Hom}_G(F_{-(p+*)}, H_{-p}(K_*)) \rightarrow 0$$

is exact.

Then, considering the two respective induced long exact sequences in homology, and using the five lemma, we obtain that, for all $p, q \in \mathbb{Z}$,

$$E_{p,q}^1 = H_{p+q}(E_{p, *-p}^0) \cong H_{p+q}(\text{Hom}_G(F_{-(p+*)}, H_{-p}(K_*))) = H^{-2p-q}(G, H_{-p}(K_*)).$$

Thereafter, the spectral sequence E and the Hochschild-Serre spectral sequence are naturally isomorphic (modulo reindexing) because of the naturality of the last isomorphism and because the differential of the two spectral sequences are induced by the same morphisms. \square

We end this subsection by an essential point : the functor L commutes with the operation that associates to a cubical diagram in \mathcal{C}^G (resp. \mathcal{C}_-) its simple filtered complex. For the definition of a cubical diagram and the associated simple filtered complex, see [12] section 1.

Proposition 3.10. *Let \mathcal{K} be a cubical diagram of type \square_n^+ in \mathcal{C}^G , then $\mathbf{s}(L_*(\mathcal{K})) = L_*(\mathbf{s}\mathcal{K})$ (in \mathcal{C}_- if we consider the same projective resolution).*

Proof. Direct computation. \square

3.2 Equivariant weight complex

Applying the functor L to the weight complex with action, we obtain the equivariant weight complex which will induce the equivariant weight filtration on the equivariant homology we define below.

Definition 3.11. Let X be a real algebraic G -variety. We denote $C_*^G(X) := L_*^G(C_*(X))$ and, for $n \in \mathbb{Z}$, we associate to X

$$H_n(X; G) := H_n(G, C_*(X)) = H_n(C_*^G(X)),$$

its n^{th} equivariant homology group, where $C_*(X)$ is the G -complex of semialgebraic chains with closed supports of the set of real points of X .

Remark 3.12. • The Hochschild-Serre spectral sequence

$${}_I E_{p,q}^2 = H^{-p}(G, H_q(X)) \Rightarrow H_{p+q}(X; G)$$

allows us to take a grasp of this equivariant homology from a geometric point of view. It takes into account the geometry of the real points of the considered real algebraic G -variety, the geometry of the action of G on it, and the structure of the group as well.

- This equivariant homology is the same as J. van Hamel's equivariant homology defined in [15] Chapter III Definition 1.2 (at least for compact G -real algebraic varieties).
- Pour $G = \{e\}$, $H_n(X; G) = H_n(X)$.

Example 3.13. We use the Hochschild-Serre spectral sequence to compute the equivariant homology of the 2-dimensional sphere X , given by the equation $x^2 + y^2 + z^2 = 1$ dans \mathbb{R}^3 , equipped with the action of $G = \mathbb{Z}/2\mathbb{Z}$ given by $\sigma : (x, y) \mapsto (-x, y)$.

The page ${}_I E^2(X)$ is

$$\begin{array}{ccccc} \dots & \mathbb{Z}_2[X] & \dots & \mathbb{Z}_2[X] & \mathbb{Z}_2[X] \\ \dots & 0 & \dots & 0 & 0 \\ \dots & \mathbb{Z}_2[\{p_0\}] & \dots & \mathbb{Z}_2[\{p_0\}] & \mathbb{Z}_2[\{p_0\}] \end{array}$$

where p_0 is a point of X , that we choose in the fixed points set X^G . Then, since the differentials of the rows of the double complex inducing the spectral sequence are $1 + \sigma$, we have ${}_I E^2(X) = {}_I E^\infty(X)$ and

$$H_k(X; G) = \begin{cases} \mathbb{Z}_2[X] & \text{if } k = 1 \text{ ou } 2, \\ \mathbb{Z}_2[X] \oplus \mathbb{Z}_2[\{p_0\}] & \text{if } k \leq 0. \end{cases}$$

If we consider now the fixed point free G -action on X given by $\sigma : (x, y) \mapsto (-x, -y)$, the page ${}_I E^2(X)$ is the same (with p_0 being any point of X), but the differentials d^3 are non-trivial (we have $d^3([\{p_0\}]) = [X]$). Consequently, the Hochschild-Serre spectral sequence degenerates at the page ${}_I E^4(X)$:

$$\begin{array}{ccccc} \dots & 0 & \dots & \mathbb{Z}_2[X] & \mathbb{Z}_2[X] \\ \dots & 0 & \dots & 0 & 0 \\ \dots & 0 & \dots & 0 & 0 \end{array}$$

for this action and

$$H_k(X; G) = \begin{cases} \mathbb{Z}_2[X] & \text{if } k = 0, 1 \text{ ou } 2, \\ 0 & \text{if } k < 0. \end{cases}$$

The second spectral sequence

$${}_{II}E_{p,q}^1 = H^{-p}(G, C_q(X)) \Rightarrow H_{p+q}(X; G)$$

associated to the equivariant homology can also be useful, as in the following case :

Lemma 3.14. *Let $G = \mathbb{Z}/2\mathbb{Z}$. Then, for all real algebraic G -varieties X and all $k \in \mathbb{Z}$,*

$$H_k(X; G) = (\ker \partial_k)^G / \partial_d \left((C_{k+1}(X))^G \right) \oplus \bigoplus_{i \geq k+1} H_i(X^G).$$

Proof. The equivariant homology is the homology of the total complex associated to the double complex

$$\begin{array}{ccccc} C_d(X) & \xrightarrow{1+\sigma} & C_d(X) & \xrightarrow{1+\sigma} & C_d(X) & \rightarrow \\ \downarrow \partial_d & & \downarrow \partial_d & & \downarrow \partial_d & \\ C_{d-1}(X) & \xrightarrow{1+\sigma} & C_{d-1}(X) & \xrightarrow{1+\sigma} & C_{d-1}(X) & \rightarrow \\ \downarrow \partial_{d-1} & & \downarrow \partial_{d-1} & & \downarrow \partial_{d-1} & \\ \vdots & & \vdots & & \vdots & \\ \downarrow \partial_2 & & \downarrow \partial_2 & & \downarrow \partial_2 & \\ C_1(X) & \xrightarrow{1+\sigma} & C_1(X) & \xrightarrow{1+\sigma} & C_1(X) & \rightarrow \\ \downarrow \partial_1 & & \downarrow \partial_1 & & \downarrow \partial_1 & \\ C_0(X) & \xrightarrow{1+\sigma} & C_0(X) & \xrightarrow{1+\sigma} & C_0(X) & \rightarrow \end{array}$$

where d is the dimension of X . Using the Smith short exact sequence

$$0 \rightarrow C_*(X^G) \oplus (1 + \sigma)C_*(X) \rightarrow C_*(X) \rightarrow (1 + \sigma)C_*(X) \rightarrow 0$$

we compute the page 1 of the spectral sequence ${}_{II}E$:

$$\begin{array}{cccc} (C_d(X))^G & C_d(X^G) & C_d(X^G) & \cdots \\ \downarrow \partial_d & \downarrow \partial_d & \downarrow \partial_d & \\ (C_{d-1}(X))^G & C_{d-1}(X^G) & C_{d-1}(X^G) & \cdots \\ \downarrow \partial_{d-1} & \downarrow \partial_{d-1} & \downarrow \partial_{d-1} & \\ \vdots & \vdots & \vdots & \\ \downarrow \partial_2 & \downarrow \partial_2 & \downarrow \partial_2 & \\ (C_1(X))^G & C_1(X^G) & C_1(X^G) & \cdots \\ \downarrow \partial_1 & \downarrow \partial_1 & \downarrow \partial_1 & \\ (C_0(X))^G & C_0(X^G) & C_0(X^G) & \cdots \end{array}$$

The term ${}_{II}E^2$ is

$$\begin{array}{cccc}
(ker \partial_d)^G & H_d(X^G) & H_d(X^G) & \dots \\
(ker \partial_{d-1})^G / \partial_d((C_d(X))^G) & H_{d-1}(X^G) & H_{d-1}(X^G) & \dots \\
\vdots & \vdots & \vdots & \\
(ker \partial_1)^G / \partial_2((C_2(X))^G) & H_1(X^G) & H_1(X^G) & \dots \\
(C_0(X))^G / \partial_1((C_1(X))^G) & H_0(X^G) & H_0(X^G) & \dots
\end{array}$$

But then, the differentials ${}_{II}d^r$ for $r \geq 2$ are all trivial so the spectral sequence degenerates at the degree 2. Consequently, as ${}_{II}E$ converges to the equivariant homology of X , we have

$$H_k(X; G) = (ker \partial_k)^G / \partial_d((C_{k+1}(X))^G) \oplus \bigoplus_{i \geq k+1} H_i(X^G)$$

for all $k \in \mathbb{Z}$.

□

We finally define the equivariant weight complex. Recall that the weight complex with action has values in $H \circ \mathcal{C}^G$, as a consequence we can apply the functor $L : H \circ \mathcal{C}^G \rightarrow H \circ \mathcal{C}_-$ to it to get an element of $H \circ \mathcal{C}_-$.

Definition 3.15. *Let X be a real algebraic G -variety. We denote*

$$\Omega C_*^G(X) := L_*(\mathcal{WC}_*(X)) \in H \circ \mathcal{C}_-,$$

and we call this filtered complex the equivariant weight complex of X .

For any real algebraic G -variety X , we denote $\mathcal{F}^{can} C_*^G(X) := L_*(F^{can} C_*(X))$. As its non-equivariant analog, the equivariant filtered weight complex is an acyclic and additive extension, in $H \circ \mathcal{C}_-$, of this equivariant canonical filtration, unique up to a filtered quasi-isomorphism :

Theorem 3.16. *The operation*

$$\Omega C_*^G : \mathbf{Sch}_c^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}_- ; X \mapsto \Omega C_*^G(X)$$

is a functor, extension of the functor

$$\mathcal{F}^{can} C_*^G : \mathbf{V}^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}_- ; X \mapsto \mathcal{F}^{can} C_*^G(X),$$

and verifying the following properties :

1. *Acyclicity* : For any acyclic square (2.1) in $\mathbf{Sch}_c^G(\mathbb{R})$, the simple filtered complex of the \square_1^+ -diagram in \mathcal{C}_-

$$\begin{array}{ccc} \Omega C_*^G(\tilde{Y}) & \rightarrow & \Omega C_*^G(\tilde{X}) \\ \downarrow & & \downarrow \\ \Omega C_*^G(Y) & \rightarrow & \Omega C_*^G(X) \end{array}$$

is acyclic.

2. *Additivity* : For any equivariant closed inclusion $Y \subset X$, the simple filtered complex of the \square_0^+ -diagram in \mathcal{C}_-

$$\Omega C_*^G(Y) \rightarrow \Omega C_*^G(X)$$

is isomorphic to $\Omega C_*^G(X \setminus Y)$ in $H \circ \mathcal{C}_-$.

Furthermore, any other functor $\mathbf{Sch}_c^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}_-$ verifying these three properties is isomorphic to ΩC_*^G in $H \circ \mathcal{C}_-$, up to a unique quasi-isomorphism in \mathcal{C}_- .

Proof. The equivariant weight complex ΩC_*^G is the composition of the weight complex with action functor ${}^G\mathcal{WC}_*$ and the functor $L : H \circ \mathcal{C}^G \rightarrow H \circ \mathcal{C}_-$. Then, it is an extension of the composition of the functors $F^{can}C_* : \mathbf{V}^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}^G$ and $L : H \circ \mathcal{C}^G \rightarrow H \circ \mathcal{C}_-$.

Moreover, it verifies the acyclicity and additivity properties because so does the weight complex with action and the functor L commutes with the operation consisting in taking the simple filtered diagram of a cubical diagram.

Now, we show the unicity of the equivariant weight complex with respect to these properties, using the theorem 2.4 applied to the category \mathcal{C}_- and the functor $\mathcal{F}^{can}C_*^G : \mathbf{V}^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}_-$. Indeed,

- The category \mathcal{C}_- is a category of homological descent ([6] Propriété (1.7.5)).
- The functor $\mathcal{F}^{can}C_*^G : \mathbf{V}^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}_-$ is Φ -rectified because, fixing a projective resolution, it is defined on \mathcal{C}_- .
- It verifies condition (F1) : the functor $F^{can}C_* : \mathbf{V}^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}^G$ verifies (F1) on $H \circ \mathcal{C}^G$ and, for all complexes K, M of $H \circ \mathcal{C}^G$, we have $L(K \oplus M) = L(K) \oplus L(M)$.
- It verifies condition (F2) : the functor $F^{can}C_* : \mathbf{V}^G(\mathbb{R}) \rightarrow H \circ \mathcal{C}^G$ verifies (F2) sur $H \circ \mathcal{C}^G$ and we apply the functor L , using its commutativity with s .

□

The homology of the equivariant weight complex of a G -variety X is the equivariant homology of X . Indeed, let us consider the forgetful functor of the filtration $\mathcal{C}_- \rightarrow \mathcal{D}_-$, which induces a functor $\varphi_- : H \circ \mathcal{C}_- \rightarrow H \circ \mathcal{D}_-$. We have

$$\varphi_- \circ \Omega C_*^G = L \circ (\varphi^G \circ {}^G\mathcal{WC}_*)$$

(where $\varphi^G : H \circ \mathcal{C}^G \rightarrow H \circ \mathcal{D}^G$ is the functor induced by the forgetful functor $\mathcal{C}^G \rightarrow \mathcal{D}^G$).

But, the functor $\varphi^G \circ {}^G\mathcal{WC}_*$ is quasi-isomorphic to C_* in \mathcal{D}^G (see remark 2.5 and [13], Remarque 3.9.), then the complex $\varphi_-(\Omega C_*^G(X)) = L_*(\varphi^G({}^G\mathcal{WC}_*(X)))$ is quasi-isomorphic to $L_*(C_*(X)) = C_*^G(X)$ (the functor L preserves quasi-isomorphisms) and for all $n \in \mathbb{Z}$,

$$H_n(\Omega C_*^G(X)) = H_n(X; G).$$

Consequently, the equivariant weight complex induces a filtration on the equivariant homology of real algebraic G -varieties, called the equivariant weight filtration. We denote it by Ω .

3.3 Equivariant weight spectral sequence(s)

The equivariant weight complex induces a converging spectral sequence, that we call the equivariant weight spectral sequence and denote by $\{{}^GE^r, {}^Gd^r\}$. According to the last remark of the previous subsection, it converges to the equivariant weight filtration. We reindex it, just as in [12] section 1.3, putting

$$p' = 2p + q, \quad q' = -p, \quad r' = r + 1,$$

to get a new spectral sequence denoted by ${}^G\tilde{E}_{p',q'}^2$.

As in the non-equivariant frame, we can read the acyclicity and additivity conditions of the equivariant weight complex on it : for instance, if $Y \subset X$ is an equivariant closed inclusion, we have a long exact sequence

$$\dots \rightarrow {}^G\tilde{E}_{p,q}^2(Y) \rightarrow {}^G\tilde{E}_{p,q}^2(X) \rightarrow {}^G\tilde{E}_{p,q}^2(X \setminus Y) \rightarrow {}^G\tilde{E}_{p-1,q}^2(Y) \rightarrow \dots$$

for all $q \in \mathbb{Z}$ (proof is going the same way as in the non-equiv case : see again [12] section 1.3). Nevertheless, some significative differences between non-equivariant and equivariant weight spectral sequences appear because of the influence of the cohomology of the group G . In particular, the equivariant weight spectral sequence is no longer left bounded (as a consequence, the additivity spectral sequences are not bounded either), and, on the compact nonsingular varieties, it does not degenerate on page 2 anymore (3.19, 3.20, 3.23).

The richness of the group action on the variety provides much information and the study of the sole equivariant weight spectral sequence is not enough to grasp it all. However, looking deeper at two other spectral sequences, constituting the elementary bricks of the first one, will allow us to have a better understanding of the equivariant weight filtration and to extract some (bounded) additivity.

The property that we use to construct these new spectral sequences is the following one :

Proposition 3.17. *For all p, q in \mathbb{Z} ,*

$${}^G\tilde{E}_{p,q}^2 = H_p(G, \tilde{E}_{*,q}^1).$$

Proof. : We have ${}^GE_{p,q}^1 = H_{p+q}({}^GE_{p,*-p}^0)$ and

$${}^GE_{p,*-p}^0 = E_{p,*-p}^0(L_*({}^G\mathcal{WC}_*(X))) = L_*\left(E_{p,*-p}^0({}^G\mathcal{WC}_*(X))\right)$$

then ${}^GE_{p,q}^1 = H_{p+q}(G, E_{p,*-p}^0)$. □

Saying that the term ${}^G\tilde{E}^2$ can express as the homology of the group G with coefficients in the non-equivariant weight spectral sequence allows us to consider the two associated spectral sequences

$$\begin{aligned} {}^qI E_{\alpha,\beta}^2 &= H^{-\alpha} \left(G, \tilde{E}_{\beta,q}^2 \right) \\ {}^qI E_{\alpha,\beta}^1 &= H^{-\alpha} \left(G, \tilde{E}_{\beta,q}^1 \right) \end{aligned}$$

which converge both to $H_{\alpha+\beta} \left(G, \tilde{E}_{*,q}^1 \right) (= {}^G\tilde{E}_{\alpha+\beta,q}^2)$.

Remark 3.18. The spectral sequence ${}^qI E$ depends on the considered representative in \mathcal{C}^G of the weight complex (with action), contrary to the spectral sequence qE .

Let X be a real algebraic G -variety of dimension d . As a particular case of these last facts, we can look at the terms of the d^{th} row of the reindexed equivariant weight spectral sequence :

Corollary 3.19. *For all $p \in \mathbb{Z}$,*

$${}^G\tilde{E}_{p,d}^2 = H^{-p} \left(G, \tilde{E}_{0,d}^2 \right)$$

Proof. We consider the spectral sequence

$${}^dI E_{\alpha,\beta}^2 = H^{-\alpha} \left(G, \tilde{E}_{\beta,d}^2 \right) \Rightarrow {}^G\tilde{E}_{\alpha+\beta,d}^2,$$

We have $\tilde{E}_{\beta,d}^2 = 0$ if $\beta \neq 0$ ([12] section 1.3) so the spectral sequence converges at ${}^dI E^2$ and

$${}^G\tilde{E}_{p,d}^2 = \bigoplus_{\alpha+\beta=p} {}^dI E_{\alpha,\beta}^2 = H^{-p} \left(G, \tilde{E}_{0,d}^2 \right).$$

□

In particular, we notice that, in the general case, there is an infinity of non-zero terms in the row $q = d$ of the reindexed equivariant spectral sequence. This is also the case for the other rows, that is, they are not left bounded :

Corollary 3.20. *For all $r \geq 2$, $p, q \in \mathbb{Z}$, if ${}^G\tilde{E}_{p,q}^r \neq 0$ then $0 \leq q \leq d$ and $p + q \leq d$.*

Proof. For all $q \in \mathbb{Z}$, we have the spectral sequence

$${}^qI E_{\alpha,\beta}^2 = H^{-\alpha} \left(G, \tilde{E}_{\beta,q}^2 \right) \Rightarrow {}^G\tilde{E}_{\alpha+\beta,q}^2.$$

If $\alpha > 0$, $H^{-\alpha}(G, \cdot) = 0$ and, according to [12] section 1.3., for all $\beta \in \mathbb{Z}$, if $\tilde{E}_{\beta,q}^2 \neq 0$ then $\beta \geq 0$, $q \geq 0$ and $\beta + q \leq d$. But, if $p + q > d$, for all $\alpha, \beta \in \mathbb{Z}$ such that $\alpha + \beta = p$, we have $\alpha + \beta + q \geq d$, and either $\beta + q > d$ or $\alpha > 0$. □

Nevertheless, the equivariant weight filtration on the equivariant homology is bounded but, contrary to the non-equivariant weight filtration, the first degree on the left which is always zero is only depending on the dimension of the considered variety for all homology spaces.

Corollary 3.21. *The equivariant weight filtration on the equivariant homology of X is a bounded increasing filtration*

$$0 = \Omega_{-d-1}H_k(X; G) \subset \Omega_{-d}H_k(X; G) \subset \dots \subset \Omega_0H_k(X; G) = H_k(X; G).$$

Proof. We prove $\Omega_0H_k(X; G) = H_k(X; G)$ and $\Omega_{-d-1}H_k(X; G) = 0$ using the equalities

$$\Omega_pH_k(X; G) = \bigoplus_{q \geq 0} {}^G\tilde{E}_{k+p-q, -(p-q)}^\infty$$

and the previous corollary 3.20. □

Remark 3.22. The fact that the weight spectral sequence was left-bounded was a key tool to extract additive invariants from its page 2, namely the virtual Betti numbers (see [12] section 1.3). Because this is not the case in the equivariant frame, the equivariant weight spectral sequence's additivity long exact sequences are not finite and furthermore, the condition of compactness-nonsingularity does not imply the convergence of the equivariant weight spectral sequence on page 2 in general.

Proposition 3.23. *Assume X to be compact and nonsingular. Then the equivariant weight spectral sequence ${}^G\tilde{E}$ of X coincides, from page 2, with the Hochschild-Serre spectral sequence*

$${}_pE_{p,q}^2(X) = H^{-p}(G, H_q(X)) \Rightarrow H_{p+q}(X; G)$$

associated to X and G .

Proof. If X is compact nonsingular, the weight complex with action ${}^G\mathcal{WC}_*(X)$ is quasi-isomorphic to $F^{can}C_*(X)$ in \mathcal{C}^G (remark 2.5). The functor L preserves filtered quasi-isomorphisms, so the equivariant weight complex ΩC_*^G is quasi-isomorphic to $\mathcal{F}^{can}C_*^G = \mathcal{F}^{can}L_*(C_*)$ in \mathcal{C}_- on compact nonsingular G -real algebraic varieties (it goes the same for any realization of the equivariant weight complex in \mathcal{C}^G).

Then, we use lemma 3.9 to say that, after reindexing, the equivariant weight spectral sequence of X is isomorphic to the Hochschild-Serre spectral sequence of X from page 2. □

In particular, even in the case of a compact nonsingular variety, the equivariant weight spectral sequence may not converge on page 2 : see example 3.13.

Below, we compute the equivariant weight spectral sequences and filtrations of a singular real algebraic variety equipped with two different algebraic involutions.

Example 3.24. Let X be the real algebraic curve given by the equation $y^2 = x^2 - x^4$ in \mathbb{R}^2 .

1. We consider the action of $G = \mathbb{Z}/2\mathbb{Z}$ on X given by $\sigma : (x, y) \mapsto (-x, y)$. The $\tilde{E}^2(X)$ page of the reindexed weight spectral sequence is given by

$$\begin{array}{c} \mathbb{Z}_2[X] \\ \mathbb{Z}_2[\{p_0\}] \quad \mathbb{Z}_2[X_1] \end{array}$$

where $p_0 = (0, 0)$ is the unique fixed point of X under the action, and X_1 and X_2 are the two 1-cycles of X (here exchanged by the action).

We have

- ${}^G\tilde{E}_{p,1}^2 = H^{-p} \left(G, \tilde{E}_{0,1}^2 \right) = \mathbb{Z}_2[X]$ if $p \leq 0$, 0 otherwise.
- the terms ${}^G\tilde{E}_{p,0}^2(X)$ are given by the Hochschild-Serre spectral sequence associated to the G -complex $\tilde{E}_{*,0}^1(X)$: the page 2 writes

$$\begin{array}{ccccc} \cdots & \mathbb{Z}_2[X_1] & \cdots & \mathbb{Z}_2[X_1] & \mathbb{Z}_2[X_1] \\ \cdots & \mathbb{Z}_2[\{p_0\}] & \cdots & \mathbb{Z}_2[\{p_0\}] & \mathbb{Z}_2[\{p_0\}] \end{array}$$

and since the differentials are trivial, we have

$${}^G\tilde{E}_{p,0}^2(X) = \begin{cases} \mathbb{Z}_2[X_1] & \text{if } p = 1, \\ \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[\{p_0\}] & \text{if } p \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently the page ${}^G\tilde{E}^2(X)$ is

$$\begin{array}{ccccc} \cdots & \mathbb{Z}_2[X] & \cdots & \mathbb{Z}_2[X] & \mathbb{Z}_2[X] \\ \cdots & \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[\{p_0\}] & \cdots & \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[\{p_0\}] & \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[\{p_0\}] & \mathbb{Z}_2[X_1] \end{array}$$

and the page ${}^G\tilde{E}^3(X)$ writes

$$\begin{array}{ccccc} \cdots & 0 & \cdots & 0 & \mathbb{Z}_2[X] \\ \cdots & \mathbb{Z}_2[\{p_0\}] & \cdots & \mathbb{Z}_2[\{p_0\}] & \mathbb{Z}_2[\{p_0\}] & 0 \end{array}$$

because ${}^G\tilde{d}^2([X_1]) = \partial \oplus (1 + \sigma)([X_1]) = [X]$ and ${}^G\tilde{d}^2([\{p_0\}]) = 0$.

Consequently, the equivariant weight spectral sequence ${}^G\tilde{E}(X)$ converges at ${}^G\tilde{E}^3(X)$ and the equivariant weight filtration of X with respect to the action σ of G is given by

$$\Omega_{-1}H_1(X; G) = \Omega_0H_1(X; G) = \mathbb{Z}_2[X]$$

and

$$0 = \Omega_{-1}H_k(X; G) \subset \Omega_0H_k(X; G) = \mathbb{Z}_2[\{p_0\}]$$

for $k \leq 0$.

2. If we consider now the action of G given by $(x, y) \mapsto (x, -y)$, going the same way, we obtain the same page ${}^G\tilde{E}^2(X)$

$$\begin{array}{ccccc} \cdots & \mathbb{Z}_2[X] & \cdots & \mathbb{Z}_2[X] & \mathbb{Z}_2[X] \\ \cdots & \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[\{p_0\}] & \cdots & \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[\{p_0\}] & \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[\{p_0\}] & \mathbb{Z}_2[X_1] \end{array}$$

But here, ${}^G\tilde{d}^2([X_1]) = 0$ because the cycle X_1 is globally invariant under the action. As a consequence, the differential ${}^G\tilde{d}^2$ is trivial and the equivariant spectral sequence of X converges at ${}^G\tilde{E}^2(X)$. The equivariant weight filtration on the equivariant homology of X is then given by

$$\mathbb{Z}_2[X] = \Omega_{-1}H_1(X; G) \subset \Omega_0H_1(X; G) = \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[X_2]$$

et

$$\mathbb{Z}_2[X] = \Omega_{-1}H_k(X; G) \subset \Omega_0H_k(X; G) = \mathbb{Z}_2[X_1] \oplus \mathbb{Z}_2[X_2] \oplus \mathbb{Z}_2[\{p_0\}]$$

for $k \leq 0$.

3.4 The odd-order group case

We just saw that, in general, the action of G complicates the excavation of additive invariants from the equivariant weight spectral sequence. Though, if the order of G is odd, the equivariant weight objects correspond to the invariants of the non-equivariant objects under the action of G , allowing us to extract additive invariants easily. This is based on the fact that our chains have coefficients in \mathbb{Z}_2 : if the order of G is odd, the ring $\mathbb{Z}_2[G]$ is semi-simple according to Maschke theorem (see for instance [2] Theorem 2.1.1).

Consequently, every $\mathbb{Z}_2[G]$ -module is projective (and injective). Then a projective resolution of \mathbb{Z}_2 on $\mathbb{Z}_2[G]$ is simply given by

$$\dots \rightarrow 0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 0, \quad (3.1)$$

and the cohomology of G with coefficients in a $\mathbb{Z}_2[G]$ -module M is $H^n(G, M) = \text{Hom}_{k[G]}(k, M) = M^G$ for $n = 0$, and 0 otherwise.

Furthermore, the semi-simplicity of $\mathbb{Z}_2[G]$ is equivalent to the condition that every short exact sequence of $\mathbb{Z}_2[G]$ -modules is split. In particular, the functor Γ^G that associates to any $\mathbb{Z}_2[G]$ the set of its invariant elements is exact.

Let G be a group of odd order. Taking the trivial resolution (3.1) of \mathbb{Z}_2 as a projective resolution on $\mathbb{Z}_2[G]$, we have that the double complex associated to the functor L is reduced to the sole non-zero column $p = 0$, and then, for any G -complex K_* ,

$$L_*(K_*) = (K_*)^G$$

and

$$H_*(G, K_*) = H_n(L_*(K_*)) = H_n((K_*)^G) = (H_n(K_*))^G,$$

(the functor Γ^G is exact on the category of $\mathbb{Z}_2[G]$ -modules).

Finally, the spectral sequences ${}_IE$ et ${}_{II}E$ associated to a G -complex K_* coincide and converge at page 2 :

$${}_IE_{p,q}^2 = H^{-p}(G, H_q(K_*)) = \begin{cases} (H_q(K_*))^G = H_q((K_*)^G) & \text{if } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$${}_{II}E_{p,q}^2 = H_q(H^{-p}(G, K_*)) = \begin{cases} H_q((K_*)^G) = (H_q(K_*))^G & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, if we consider a G -real algebraic variety X , we have

$$H_*(X; G) = H_*((C_*(X))^G) = (H_*(X))^G, \quad \Omega C_*^G(X) = (\mathcal{W}C_*(X))^G, \quad {}^GE(X) = (E(X))^G.$$

In particular, we can surround the non-zero terms of the reindexed equivariant spectral sequence of X into the triangle of vertices $(0, 0)$, $(0, d)$ and $(d, 0)$ (if d is the dimension of X). Consequently, we are able to recover the equivariant virtual Betti numbers (see [5] and section 4.1 below) in case of the group G being of odd order.

Proposition 3.25. *Let G be an odd-order finite group. For all real algebraic G -varieties X and all $q \in \mathbb{Z}$, the q^{th} equivariant virtual Betti number of X ([5]) is the alternating sum of the dimensions of the row q terms of the equivariant weight spectral sequence :*

$$\beta_q^G(X) = \sum_p (-1)^p \dim_{\mathbb{Z}_2} {}^G\widetilde{E}_{p,q}^2.$$

Proof. In this frame, the (q^{th}) additivity long exact sequence for an equivariant closed inclusion is finite and gives us the additivity of the β_q^G 's. If X is compact nonsingular, ${}^G\widetilde{E}_{p,q}^2 = \left(\widetilde{E}_{p,q}^2\right)^G = 0$ if $p \neq 0$ and

$${}^G\widetilde{E}_{0,q}^2 = \left(\widetilde{E}_{0,q}^2\right)^G = (H_q(X))^G = H_q(X; G).$$

□

4 Additive invariants

4.1 Additivity

Let G be a finite group.

In this section, we realize the equivariant weight complex through the Nash-constructible filtration. This will allow us to find some additivity on one of the spectral sequences converging to the equivariant spectral sequence (4.2). From this, we then define some additive invariants B_k^G on real algebraic G -varieties (4.4). They are actually the Euler characteristic of some other spectral sequences ${}^k\widetilde{E}$, constructed from the equivariant geometry embodied by the Nash-constructible filtration (4.9).

If $G = \mathbb{Z}/2\mathbb{Z}$, we compute the spectral sequences ${}^k\widetilde{E}$ and obtain a formula (4.11) that allows us to prove that the B_k^G 's coincide with G. Fichou's equivariant virtual Betti numbers ([5]) in some cases (4.14). These are the unique additive invariants defined on the category of real algebraic G -varieties that equal the dimensions of equivariant homology groups on compact nonsingular varieties.

Definition 4.1. *For X a real algebraic G -variety, we denote by $\Lambda C_*^G(X)$ the complex $L(\mathcal{NC}_*(X))$ of \mathcal{C}_- (choosing a projective resolution of \mathbb{Z} or \mathbb{Z}_2 on $\mathbb{Z}[G]$, resp. $\mathbb{Z}_2[G]$), that we call the equivariant Nash-constructible complex. Since the Nash-constructible filtration with action realizes the weight complex with action (2.5), the functor*

$$\Lambda C_*^G : \mathbf{Sch}_c^G(\mathbb{R}) \rightarrow \mathcal{C}_- ; X \mapsto \Lambda C_*^G(X)$$

realizes the equivariant weight complex (in $H \circ \mathcal{C}_-$).

The realization of the equivariant weight complex at the chain level leads us to find an additivity that we could not read directly on the equivariant weight spectral sequence :

Proposition 4.2. *Let $Y \subset X$ be an equivariant closed inclusion in $\mathbf{Sch}_{\mathbb{C}}^G(\mathbb{R})$. For any q and i in \mathbb{Z} , there is a finite long exact sequence*

$$\cdots \rightarrow {}^q E_{i,j}^2(Y) \rightarrow {}^q E_{i,j}^2(X) \rightarrow {}^q E_{i,j}^2(X \setminus Y) \rightarrow {}^q E_{i,j-1}^2(Y) \rightarrow \cdots,$$

where ${}^q E_{i,j}^2 = H_j \left(H^{-i} \left(G, \widetilde{E}_{*,q}^1 \right) \right)$ describes the page 2 of the spectral sequence ${}^q E$ (3.17 and following), computed from the Nash-constructible filtration, which converges to the q^{th} row of the term ${}^G \widetilde{E}^2$ of the equivariant weight spectral sequence.

Proof. We have ([12] Proof of Theorem 3.6. and [13] Remarque 3.9.) equivariant short exact sequences

$$0 \rightarrow \mathcal{N}_p C_k(Y) \rightarrow \mathcal{N}_p C_k(X) \rightarrow \mathcal{N}_p C_k(X \setminus Y) \rightarrow 0$$

which are split by the closure morphism $c \in \mathcal{N}_p C_k(X \setminus Y) \mapsto \bar{c} \in \mathcal{N}_p C_k(X)$ (it is an equivariant morphism by [13] Proposition 2.4.), and induce equivariant split short exact sequences

$$0 \rightarrow E_{p,q}^0(Y) \rightarrow E_{p,q}^0(X) \rightarrow E_{p,q}^0(X \setminus Y) \rightarrow 0.$$

Fix p and q . There is a group cohomology long exact sequence :

$$\begin{aligned} 0 \rightarrow (E_{p,q}^0(Y))^G \rightarrow (E_{p,q}^0(X))^G \rightarrow (E_{p,q}^0(X \setminus Y))^G \rightarrow \\ \rightarrow H^1(G, E_{p,q}^0(Y)) \rightarrow H^1(G, E_{p,q}^0(X)) \rightarrow H^1(G, E_{p,q}^0(X \setminus Y)) \rightarrow \dots \end{aligned}$$

But the (equivariantly) split morphism $E_{p,q}^0(X) \rightarrow E_{p,q}^0(X \setminus Y)$ induces split morphisms $H^k(G, E_{p,q}^0(X)) \rightarrow H^k(G, E_{p,q}^0(X \setminus Y))$ for all $k \in \mathbb{Z}$, which are in particular surjective.

Then, the short sequences of complexes

$$0 \rightarrow H^k(G, E_{p,*}^0(Y)) \rightarrow H^k(G, E_{p,*}^0(X)) \rightarrow H^k(G, E_{p,*}^0(X \setminus Y)) \rightarrow 0$$

are exact for all k and p in \mathbb{Z} , as well as the short sequences

$$0 \rightarrow H^{-i}(G, E_{-q,*+2q}^0(Y)) \rightarrow H^{-i}(G, E_{-q,*+2q}^0(X)) \rightarrow H^{-i}(G, E_{-q,*+2q}^0(X \setminus Y)) \rightarrow 0$$

for all i and q . Taking the long exact sequences in homology and reindexing, we obtain the desired result. □

Remark 4.3. The equivariantly split short exact sequences

$$0 \rightarrow \mathcal{N}_p C_k(Y) \rightarrow \mathcal{N}_p C_k(X) \rightarrow \mathcal{N}_p C_k(X \setminus Y) \rightarrow 0$$

induce also the long exact homology sequences of the pairs $((\mathcal{N}_p C_*)^G, (\mathcal{N}_p C_*)^G)$ for all $p \in \mathbb{Z}$:

$$H_k \left(\frac{(\mathcal{N}_p C_*(Y))^G}{(\mathcal{N}_{p-1} C_*(Y))^G} \right) \rightarrow H_k \left(\frac{(\mathcal{N}_p C_*(X))^G}{(\mathcal{N}_{p-1} C_*(X))^G} \right) \rightarrow H_k \left(\frac{(\mathcal{N}_p C_*(X \setminus Y))^G}{(\mathcal{N}_{p-1} C_*(X \setminus Y))^G} \right) \rightarrow H_{k-1} \left(\frac{(\mathcal{N}_p C_*(Y))^G}{(\mathcal{N}_{p-1} C_*(Y))^G} \right)$$

Based on the finite long exact sequences of proposition 4.2, for all q and i in \mathbb{Z} we define

$${}^qB_i(X) := \sum_j (-1)^j \dim_{\mathbb{Z}_2} {}^qE_{i,j}^2(X)$$

for any real algebraic G -variety X . Each of these additive invariants is constructed from the very geometry of real algebraic G -varieties -geometry of the object (embodied by the Nash-constructible filtration) and geometry of the action- up to some degree and dimension depending on q and i .

Now, taking diagonal sums of the qB_i 's, we obtain other invariants B_k^G for all k in \mathbb{Z} . We will give some geometric meaning about these in 4.9, realizing each of them as the Euler characteristic of a spectral sequence.

Definition 4.4. *For every real algebraic G -variety X , for all $k \in \mathbb{Z}$, we put*

$$B_k^G(X) := \sum_{q+i=k} {}^qB_i(X).$$

Remark 4.5. We can define ${}^qB_i(X)$ and $B_k^G(X)$ only if the involved spaces ${}^qE_{i,j}^2(X) = H_j \left(H^{-i} \left(G, \tilde{E}_{*,q}^1(X) \right) \right)$ are finite-dimensional. In future work, the next important step will be to decide if this is indeed the case for any real algebraic G -variety and any index.

This question should be paired with the grasp of the complexes $(\mathcal{N}_\alpha C_*(X))^G$ and their homologies, which arise regularly during our present work (3.14, 4.11). In particular, we will see in subsection 4.3 how, in the case $G = \mathbb{Z}/2\mathbb{Z}$, these invariant chains come out into the spectral sequence we construct to recover in a geometric way the equivariant invariants B_k^G .

This study should require even more geometric understanding of the semi-algebraic chains with action, and the use of equivariant subtle tools and techniques on these.

Consequently, in the following, we will restrict to the cases of varieties for which the terms ${}^qE_{i,j}^2$ are finite-dimensional, keeping in mind that in future work, we plan to show that this assumption is automatically verified for all real algebraic G -varieties.

Theorem 4.6. *The invariants $B_k^G(\cdot)$ are additive on real algebraic G -varieties on which they are well-defined.*

Proof. Consequence of 4.2. □

Remark 4.7. It should be stressed that the additivity of these invariants strongly depends on the use of the Nash-constructible filtration to realize the different weight complexes.

Example 4.8. 1. Let X be the real algebraic curve defined by the equation $y^2 = x^2 - x^4$ in \mathbb{R}^2 and consider the $\mathbb{Z}/2\mathbb{Z}$ -action given by $(x, y) \mapsto (-x, y)$. Keeping the notations of 3.24, we have

$${}^qE_{i,j}^2(X) = \begin{cases} \mathbb{Z}_2[X] & \text{if } q = 1, i \leq 0 \text{ and } j = 0, \\ \mathbb{Z}_2[X_1] & \text{if } q = 0, i \leq 0 \text{ and } j = 1, \\ \mathbb{Z}_2[\{p_0\}] & \text{if } q = 0, i \leq 0 \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$${}^qB_i(X) = \sum_j (-1)^j \dim_{\mathbb{Z}_2} {}^qE_{i,j}^2(X) = \begin{cases} 1 & \text{if } q = 1 \text{ and } i \leq 0, \\ 0 & \text{if } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_k^G(X) = \sum_{q+i=k} = \begin{cases} 1 & \text{if } k \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

2. Take the same variety X and equip it with the action $(x, y) \mapsto (x, -y)$. Here we have

$${}^qE_{i,j}^2(X) = \begin{cases} \mathbb{Z}_2[X] & \text{if } q = 1, i \leq 0 \text{ and } j = 0, \\ \mathbb{Z}_2[X_1] & \text{if } q = 0, i = 0 \text{ and } j = 1, \\ 0 & \text{if } q = 0, i < 0 \text{ and } j = 1, \\ \mathbb{Z}_2[\{p_0\}] \oplus \mathbb{Z}_2[\{p_1\}] & \text{if } q = 0, i \leq 0 \text{ and } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where p_1 is one of the two invariant points under this action other than p_0 . Therefore, we have

$${}^qB_i(X) = \begin{cases} 1 & \text{if } q = 1 \text{ and } i \leq 0, \\ 1 & \text{if } q = 0 \text{ and } i = 0, \\ 2 & \text{if } q = 0 \text{ and } i < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_k^G(X) = \begin{cases} 1 & \text{if } k = 1, \\ 2 & \text{if } k = 0, \\ 3 & \text{if } k < 0, \\ 0 & \text{otherwise,} \end{cases}$$

In the next subsection 4.2, in order to understand the invariants B_k^G in a geometrical way, we realize each of them as the Euler characteristic of a spectral sequence, constructed from the equivariant geometry of G -real algebraic varieties. Precisely, for each $k \in \mathbb{Z}$, this spectral sequence will be induced from a double complex ${}^k\widehat{C}$, arised itself from short exact sequences induced by the Nash-constructible filtration (with action) and long exact sequences induced by the cohomology of the group G .

4.2 The double complex \widehat{C} and the spectral sequence \widehat{E}

We construct a double complex in a more general frame, using the Ext functor from which group cohomology is a particular case (see the beginning of subsection 3.1).

Let \mathcal{A} be an abelian category on \mathbb{Z}_2 . We take $M \in \mathcal{A}$ and K_* a bounded chain complex in \mathcal{A} , equipped with a bounded increasing filtration. For all $k \in \mathbb{Z}$, we construct a double complex

$${}^k\widehat{C}(M, FK_*) := \left(\text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \right) \right)_{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}} =$$

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \downarrow \\ \longrightarrow & \text{Ext}_{\mathcal{A}}^{-k-(\alpha+1)} \left(M, \frac{F_{\alpha+1}K_{\beta+1}}{F_{\alpha}K_{\beta+1}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta+1}}{F_{\alpha-1}K_{\beta+1}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta+1}}{F_{\alpha-2}K_{\beta+1}} \right) & \longrightarrow \\ & \downarrow d^1 & & \downarrow d^1 & & \downarrow d^1 & \\ \longrightarrow & \text{Ext}_{\mathcal{A}}^{-k-(\alpha+1)} \left(M, \frac{F_{\alpha+1}K_{\beta}}{F_{\alpha}K_{\beta}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta}}{F_{\alpha-2}K_{\beta}} \right) & \longrightarrow \\ & \downarrow d^1 & & \downarrow d^1 & & \downarrow d^1 & \\ \longrightarrow & \text{Ext}_{\mathcal{A}}^{-k-(\alpha+1)} \left(M, \frac{F_{\alpha+1}K_{\beta-1}}{F_{\alpha}K_{\beta-1}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta-1}}{F_{\alpha-1}K_{\beta-1}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta-1}}{F_{\alpha-2}K_{\beta-1}} \right) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

Let $k \in \mathbb{Z}$. The differential

$$d^1 : \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \right) \longrightarrow \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta-1}}{F_{\alpha-1}K_{\beta-1}} \right)$$

is induced from the functoriality of $\text{Ext}_{\mathcal{A}}^{-k-\alpha}(M, \cdot)$, and the differential

$$d^0 : \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \right) \rightarrow \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta}}{F_{\alpha-2}K_{\beta}} \right)$$

is given by the commutative (by definition) diagram

$$\begin{array}{ccc} \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta}}{F_{\alpha-2}K_{\beta}} \right) & \xrightarrow{\quad} & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta}}{F_{\alpha-2}K_{\beta}} \right) \\ \uparrow & \nearrow d^0 & \\ \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \right) & & \\ \uparrow & \nwarrow d^0 & \\ \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, F_{\alpha}K_{\beta} \right) & \longleftarrow & \text{Ext}_{\mathcal{A}}^{-k-(\alpha+1)} \left(M, \frac{F_{\alpha+1}K_{\beta}}{F_{\alpha}K_{\beta}} \right) \end{array}$$

The morphisms $\text{Ext}_{\mathcal{A}}^{-k-(\alpha+1)} \left(M, \frac{F_{\alpha+1}K_{\beta}}{F_{\alpha}K_{\beta}} \right) \rightarrow \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, F_{\alpha}K_{\beta} \right)$ are connecting morphisms from the long exact sequences ([2], [1])

$$\cdots \rightarrow \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, F_{\alpha-1}K_{\beta} \right) \rightarrow \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, F_{\alpha}K_{\beta} \right) \rightarrow \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \right) \rightarrow$$

$$\rightarrow \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)}(M, F_{\alpha-1}K_{\beta}) \rightarrow \dots$$

induced from the short exact sequences

$$0 \rightarrow F_{\alpha-1}K_{\beta} \rightarrow F_{\alpha}K_{\beta} \rightarrow \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \rightarrow 0.$$

The fact that d_1 and d_0 commute comes from the following diagram

$$\begin{array}{ccccc} \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta}}{F_{\alpha-1}K_{\beta}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta}}{F_{\alpha-2}K_{\beta}} \right) & & \\ & \searrow & \nearrow & & \\ & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} (M, F_{\alpha-1}K_{\beta}) & & & \\ & \downarrow & & & \\ & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} (M, F_{\alpha-1}K_{\beta-1}) & & & \\ & \nearrow & \searrow & & \\ \text{Ext}_{\mathcal{A}}^{-k-\alpha} \left(M, \frac{F_{\alpha}K_{\beta-1}}{F_{\alpha-1}K_{\beta-1}} \right) & \xrightarrow{d^0} & \text{Ext}_{\mathcal{A}}^{-k-(\alpha-1)} \left(M, \frac{F_{\alpha-1}K_{\beta-1}}{F_{\alpha-2}K_{\beta-1}} \right) & & \\ \downarrow d^1 & & \downarrow d^1 & & \end{array}$$

which is commutative, thanks to the naturality property of the long exact sequence induced by $\text{Ext}_{\mathcal{A}}(M, \cdot)$ ([2], [1]).

Since the complex K_* and its filtration F are bounded, the double complex ${}^k\widehat{C}(M, FK_*)$ is bounded and the two spectral sequences it induces converge to the homology of the associated total complex.

Now, take \mathcal{A} to be the category of $\mathbb{Z}_2[G]$ -modules, $M := \mathbb{Z}_2$ and $FK_* := \mathcal{N}C_*(X)$, where X is a real algebraic G -variety. For all $k \in \mathbb{Z}$, we denote simply by ${}^k\widehat{C}(X)$ the double complex ${}^k\widehat{C}(\mathbb{Z}_2, \mathcal{N}C_*(X))$, and we have

$${}^k\widehat{C}_{\alpha,\beta}(X) = H^{-k-\alpha} \left(G, \frac{\mathcal{N}_{\alpha}C_{\beta}(X)}{\mathcal{N}_{\alpha-1}C_{\beta}(X)} \right).$$

The term ${}_I^k\widehat{E}^1$ of the “second” induced spectral sequence is then given by

$${}_I^k\widehat{E}_{\alpha,\beta}^1 = H_{\beta} \left(H^{-k-\alpha} \left(G, \frac{\mathcal{N}_{\alpha}C_*(X)}{\mathcal{N}_{\alpha-1}C_*(X)} \right) \right).$$

Theorem 4.9. *Let k be in \mathbb{Z} . When all the terms ${}_I^k\widehat{E}_{\alpha,\beta}^1$ are finite-dimensional, we have*

$$\chi \left({}_I^k\widehat{E}^1 \right) = B_k^G(X).$$

Proof. For all q and i such that $q + i = k$, we have

$$\begin{aligned}
{}^q B_i &= \sum_j (-1)^j \dim_{\mathbb{Z}_2} {}^q I I E_{i,j}^2(X) \\
&= \chi \left(H_* \left(H^{-i} \left(G, \frac{\mathcal{N}_{-q} C_{*+q}}{\mathcal{N}_{-q-1} C_{*+q}} \right) \right) \right) \\
&= (-1)^q \chi \left(H_* \left(H^{-i} \left(G, \frac{\mathcal{N}_{-q} C_*}{\mathcal{N}_{-q-1} C_*} \right) \right) \right).
\end{aligned}$$

Thus

$$\begin{aligned}
B_k^G(X) &= \sum_{q+i=k} {}^q B_i \\
&= \sum_{q+i=k} (-1)^q \chi \left(H_* \left(H^{-i} \left(G, \frac{\mathcal{N}_{-q} C_*}{\mathcal{N}_{-q-1} C_*} \right) \right) \right) \\
&= \sum_{\alpha} (-1)^{\alpha} \chi \left(H_* \left(H^{-k-\alpha} \left(G, \frac{\mathcal{N}_{\alpha} C_*}{\mathcal{N}_{\alpha-1} C_*} \right) \right) \right) \\
&= \chi \left({}^k I I \widehat{E}^1 \right)
\end{aligned}$$

□

Since the Euler characteristic of a spectral sequence does not depend on the page where we compute it (at least from the first page where it is well-defined), we simply write

$$B_k^G(X) = \chi \left({}^k I I \widehat{E} \right).$$

Furthermore, it is equal to the Euler characteristic of the “first” spectral sequence induced by the double complex ${}^k \widehat{C}$ (through the Euler characteristic of the homology of the total complex the two spectral sequences converge to) :

$$B_k^G(X) = \chi \left({}^k I I \widehat{E} \right).$$

These realizations of each B_k^G are of great interest. It shows that these invariants come from the very equivariant geometry of real algebraic G -varieties, an information held by these spectral sequences and their computation.

In the next subsection, we focus on the case $G = \mathbb{Z}/2\mathbb{Z}$ and compute the spectral sequences ${}^k I I \widehat{E}$, obtaining in particular a formula for the B_k^G 's of real algebraic varieties equipped with an algebraic involution.

4.3 The case $G = \mathbb{Z}/2\mathbb{Z}$

Let $G := \mathbb{Z}/2\mathbb{Z}$.

Even the action of the smallest non-trivial group enriches considerably the structures of G -real algebraic varieties (3.24), even compact nonsingular ones (3.13). The consideration of

algebraic involutions already makes more difficult the extraction of the equivariant information from the equivariant weight spectral sequence, in particular of additive invariants which, in analogy with the without-action frame, would coincide with equivariant Betti numbers on compact nonsingular varieties. However, the two spectral sequences that we just pulled out will allow us, as a new tool, to untangle all this information.

Let X be a real algebraic G -variety. We denote by σ the algebraic involution acting on X .

Let k be an integer. We are going to compute the spectral sequence ${}^k\widehat{E}(X)$, using the Smith Nash-constructible short exact sequence (2.6). We write the double complex $({}^k\widehat{C}_{\alpha,\beta}(X))_{(\alpha,\beta)\in\mathbb{Z}\times\mathbb{Z}}$:

$$\begin{array}{ccccccc}
& & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & H^{-k-(\alpha+1)} \left(G, \frac{\mathcal{N}_{\alpha+1}C_{\beta+1}}{\mathcal{N}_{\alpha}C_{\beta+1}} \right) & \xrightarrow{d^0} & H^{-k-\alpha} \left(G, \frac{\mathcal{N}_{\alpha}C_{\beta+1}}{\mathcal{N}_{\alpha-1}C_{\beta+1}} \right) & \xrightarrow{d^0} & H^{-k-(\alpha-1)} \left(G, \frac{\mathcal{N}_{\alpha-1}C_{\beta+1}}{\mathcal{N}_{\alpha-2}C_{\beta+1}} \right) & \longrightarrow \\
& \downarrow d^1 & & \downarrow d^1 & & \downarrow d^1 & \\
\longrightarrow & H^{-k-(\alpha+1)} \left(G, \frac{\mathcal{N}_{\alpha+1}C_{\beta}}{\mathcal{N}_{\alpha}C_{\beta}} \right) & \xrightarrow{d^0} & H^{-k-\alpha} \left(G, \frac{\mathcal{N}_{\alpha}C_{\beta}}{\mathcal{N}_{\alpha-1}C_{\beta}} \right) & \xrightarrow{d^0} & H^{-k-(\alpha-1)} \left(G, \frac{\mathcal{N}_{\alpha-1}C_{\beta}}{\mathcal{N}_{\alpha-2}C_{\beta}} \right) & \longrightarrow \\
& \downarrow d^1 & & \downarrow d^1 & & \downarrow d^1 & \\
\longrightarrow & H^{-k-(\alpha+1)} \left(G, \frac{\mathcal{N}_{\alpha+1}C_{\beta-1}}{\mathcal{N}_{\alpha}C_{\beta-1}} \right) & \xrightarrow{d^0} & H^{-k-\alpha} \left(G, \frac{\mathcal{N}_{\alpha}C_{\beta-1}}{\mathcal{N}_{\alpha-1}C_{\beta-1}} \right) & \xrightarrow{d^0} & H^{-k-(\alpha-1)} \left(G, \frac{\mathcal{N}_{\alpha-1}C_{\beta-1}}{\mathcal{N}_{\alpha-2}C_{\beta-1}} \right) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow &
\end{array}$$

and we have the following

Proposition 4.10.

$${}^k\widehat{E}_{\alpha,\beta}^1 = \begin{cases} \frac{\mathcal{N}_{\alpha}C_{\beta}(X^G)}{\mathcal{N}_{\alpha-1}C_{\beta}(X^G)} & \text{if } -k-\alpha \geq 1, \\ \frac{(\mathcal{N}_{\alpha}C_{\beta})^G}{(\mathcal{N}_{\alpha-1}C_{\beta})^G} & \text{if } -k-\alpha = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $-k-\alpha < 0$: The group cohomology spaces are zero in negative degree.

If $-k-\alpha = 0$: We have (3.3)

$$H^{-k-\alpha} \left(G, \frac{\mathcal{N}_{\alpha}C_{\beta}}{\mathcal{N}_{\alpha-1}C_{\beta}} \right) = \left(\frac{\mathcal{N}_{\alpha}C_{\beta}}{\mathcal{N}_{\alpha-1}C_{\beta}} \right)^G$$

and

$$H^{-k-(\alpha-1)} \left(G, \frac{\mathcal{N}_{\alpha-1}C_{\beta}}{\mathcal{N}_{\alpha-2}C_{\beta}} \right) = \frac{(\mathcal{N}_{\alpha-1}C_{\beta}/\mathcal{N}_{\alpha-2}C_{\beta})^G}{im(1+\sigma)}.$$

We denote by $[c]$ (resp. $\langle c' \rangle$) the class of a chain $c \in \mathcal{N}_\alpha C_\beta$ (resp. $c' \in \mathcal{N}_{\alpha-1} C_\beta$) in the quotient $\frac{\mathcal{N}_\alpha C_\beta}{\mathcal{N}_{\alpha-1} C_\beta}$ (resp. $\frac{\mathcal{N}_{\alpha-1} C_\beta}{\mathcal{N}_{\alpha-2} C_\beta}$), and by $\widehat{\langle c' \rangle}$ the image of an invariant class in $\frac{(\mathcal{N}_{\alpha-1} C_\beta / \mathcal{N}_{\alpha-2} C_\beta)^G}{im(1+\sigma)}$. Consider an element

$$[c] \in {}^k\widehat{E}_{\alpha,\beta}^1 = ker \left(H^{-k-\alpha} \left(G, \frac{\mathcal{N}_\alpha C_\beta}{\mathcal{N}_{\alpha-1} C_\beta} \right) \xrightarrow{d^0} H^{-k-(\alpha-1)} \left(G, \frac{\mathcal{N}_{\alpha-1} C_\beta}{\mathcal{N}_{\alpha-2} C_\beta} \right) \right),$$

that is $(1+\sigma)c \in \mathcal{N}_{\alpha-1} C_\beta$ and there exists $c_0 \in \mathcal{N}_{\alpha-1} C_\beta$ such that $\langle (1+\sigma)c \rangle = \langle (1+\sigma)c_0 \rangle$, i.e. $\gamma := (1+\sigma)(c+c_0) \in \mathcal{N}_{\alpha-2} C_\beta$.

Then γ is invariant under the action of σ , and its restriction to X^G is zero (the action commutes with the restriction, see [13] Proposition 2.4). By the exactness of the Smith Nash-constructible short sequence (2.6 and [13] Théorème 5.5), there exists $\gamma_0 \in \mathcal{N}_{\alpha-1} C_\beta$ such that $\gamma = (1+\sigma)\gamma_0$, thus $(1+\sigma)(c+c_0+\gamma_0) = 0$ and $[c] = [c+c_0+\gamma_0]$.

Consequently, we showed that the natural morphism

$$\psi : (\mathcal{N}_\alpha C_\beta)^G \rightarrow {}^k\widehat{E}_{\alpha,\beta}^1 ; c \mapsto [c]$$

is surjective. The kernel of ψ being $(\mathcal{N}_{\alpha-1} C_\beta)^G$, we have a natural isomorphism

$$\frac{(\mathcal{N}_\alpha C_\beta)^G}{(\mathcal{N}_{\alpha-1} C_\beta)^G} \cong {}^k\widehat{E}_{\alpha,\beta}^1.$$

If $-k-\alpha \geq 1$: We have

$$H^{-k-\alpha} \left(G, \frac{\mathcal{N}_\alpha C_\beta}{\mathcal{N}_{\alpha-1} C_\beta} \right) = \frac{(\mathcal{N}_\alpha C_\beta / \mathcal{N}_{\alpha-1} C_\beta)^G}{im(1+\sigma)}$$

and

$$H^{-k-(\alpha-1)} \left(G, \frac{\mathcal{N}_{\alpha-1} C_\beta}{\mathcal{N}_{\alpha-2} C_\beta} \right) = \frac{(\mathcal{N}_{\alpha-1} C_\beta / \mathcal{N}_{\alpha-2} C_\beta)^G}{im(1+\sigma)}.$$

We keep the previous notations, adding $\overline{[c]}$ to denote the image of an invariant class in $\frac{(\mathcal{N}_\alpha C_\beta / \mathcal{N}_{\alpha-1} C_\beta)^G}{im(1+\sigma)}$. Consider an element

$$\overline{[c]} \in ker \left(H^{-k-\alpha} \left(G, \frac{\mathcal{N}_\alpha C_\beta}{\mathcal{N}_{\alpha-1} C_\beta} \right) \xrightarrow{d^0} H^{-k-(\alpha-1)} \left(G, \frac{\mathcal{N}_{\alpha-1} C_\beta}{\mathcal{N}_{\alpha-2} C_\beta} \right) \right).$$

In particular,

$$[c] \in ker \left(\left(\frac{\mathcal{N}_\alpha C_\beta}{\mathcal{N}_{\alpha-1} C_\beta} \right)^G \rightarrow \frac{(\mathcal{N}_{\alpha-1} C_\beta / \mathcal{N}_{\alpha-2} C_\beta)^G}{im(1+\sigma)} \right)$$

and then, thanks to what we have done in the previous case, we know that the class of c in $\left(\frac{\mathcal{N}_\alpha C_\beta}{\mathcal{N}_{\alpha-1} C_\beta} \right)^G$ can be represented by an element c' of $(\mathcal{N}_\alpha C_\beta)^G$.

Using the exactness of the Smith Nash-constructible short sequence, there exists $\gamma' \in \mathcal{N}_{\alpha+1} C_\beta$ such that

$$c' = c'|_{X^G} + (1+\sigma)\gamma',$$

and

$$\overline{[(1+\sigma)\gamma']} \in \text{im} \left(H^{-k-(\alpha+1)} \left(G, \frac{\mathcal{N}_{\alpha+1}C_\beta}{\mathcal{N}_\alpha C_\beta} \right) \xrightarrow{d^0} H^{-k-\alpha} \left(G, \frac{\mathcal{N}_\alpha C_\beta}{\mathcal{N}_{\alpha-1}C_\beta} \right) \right).$$

Therefore, $\overline{[c]} = c'|_{X^G}$ in ${}^k\widehat{E}_{\alpha,\beta}^1$, showing that the natural morphism

$$\psi : \mathcal{N}_\alpha C_\beta(X^G) \rightarrow {}^k\widehat{E}_{\alpha,\beta}^1 ; c \mapsto \overline{[c]}$$

is surjective.

Now take an element $c \in \mathcal{N}_\alpha C_\beta(X^G)$ such that $\overline{[c]} = \overline{[(1+\sigma)c_0]}$ with $c_0 \in \mathcal{N}_{\alpha+1}C_\beta$, that is there exist $c_1 \in \mathcal{N}_\alpha C_\beta$, $c_2 \in \mathcal{N}_{\alpha-1}C_\beta$ such that

$$c = (1+\sigma)c_0 + (1+\sigma)c_1 + c_2.$$

Restricting the chains to X^G , we obtain that $c = c_2|_{X^G} \in \mathcal{N}_{\alpha-1}C_\beta(X^G)$.

Thus, the kernel of ψ is $\mathcal{N}_{\alpha-1}C_\beta(X^G)$ and we have a natural isomorphism

$$\frac{\mathcal{N}_\alpha C_\beta(X^G)}{\mathcal{N}_{\alpha-1}C_\beta(X^G)} \cong {}^k\widehat{E}_{\alpha,\beta}^1.$$

□

Consequently, the page ${}^k\widehat{E}^2$ is given, from the column $\alpha = -k$ on the left, by :

$$\begin{array}{ccccccc} H_{\beta+1} \left(\frac{(\mathcal{N}_{-k}C_*)^G}{(\mathcal{N}_{-k-1}C_*)^G} \right) & H_{\beta+1} \left(\frac{\mathcal{N}_{-k-1}C_*(X^G)}{\mathcal{N}_{-k-2}C_*(X^G)} \right) & \cdots & H_{\beta+1} \left(\frac{\mathcal{N}_\alpha C_*(X^G)}{\mathcal{N}_{\alpha-1}C_*(X^G)} \right) & \cdots \\ \\ H_\beta \left(\frac{(\mathcal{N}_{-k}C_*)^G}{(\mathcal{N}_{-k-1}C_*)^G} \right) & H_\beta \left(\frac{\mathcal{N}_{-k-1}C_*(X^G)}{\mathcal{N}_{-k-2}C_*(X^G)} \right) & \cdots & H_\beta \left(\frac{\mathcal{N}_\alpha C_*(X^G)}{\mathcal{N}_{\alpha-1}C_*(X^G)} \right) & \cdots \\ \\ H_{\beta-1} \left(\frac{(\mathcal{N}_{-k}C_*)^G}{(\mathcal{N}_{-k-1}C_*)^G} \right) & H_{\beta-1} \left(\frac{\mathcal{N}_{-k-1}C_*(X^G)}{\mathcal{N}_{-k-2}C_*(X^G)} \right) & \cdots & H_{\beta-1} \left(\frac{\mathcal{N}_\alpha C_{\beta-1}(X^G)}{\mathcal{N}_{\alpha-1}C_{\beta-1}(X^G)} \right) & \cdots \end{array}$$

Thus, the spectral sequence ${}^k\widehat{E}$ degenerates on ${}^k\widehat{E}^2$ and we have the following formula :

Proposition 4.11. *For $G = \mathbb{Z}/2\mathbb{Z}$, for all $k \in \mathbb{Z}$ and all real algebraic G -varieties X for which the following quantities are well-defined, we have*

$$B_k^G(X) = (-1)^k \chi \left(H_* \left(\frac{(\mathcal{N}_{-k}C_*)^G}{(\mathcal{N}_{-k-1}C_*)^G} \right) \right) + \sum_{q \geq k+1} \beta_q(X^G).$$

Proof. When all the spaces $H_\beta \left(\frac{(\mathcal{N}_{-k}C_*)^G}{(\mathcal{N}_{-k-1}C_*)^G} \right)$ are finite-dimensional, we have

$$\chi({}^k\widehat{E}^2) = (-1)^k \chi \left(H_* \left(\frac{(\mathcal{N}_{-k}C_*)^G}{(\mathcal{N}_{-k-1}C_*)^G} \right) \right) + \sum_{\alpha \leq -k-1} (-1)^\alpha \chi \left(H_* \left(\frac{\mathcal{N}_\alpha C_*(X^G)}{\mathcal{N}_{\alpha-1}C_*(X^G)} \right) \right).$$

But, for all $\alpha, \beta \in \mathbb{Z}$, $H_\beta \left(\frac{\mathcal{N}_\alpha C_*(X^G)}{\mathcal{N}_{\alpha-1} C_*(X^G)} \right) = \tilde{E}_{\alpha+\beta, -\alpha}^2(X^G)$ and

$$\chi \left(\tilde{E}_{\alpha+\beta, -\alpha}^2(X^G) \right) = (-1)^\alpha \beta_{-\alpha}(X^G)$$

by [12] section 1.3. □

For any G -variety X and any integer k , we then define (when it is well-defined)

$$'B_k^G(X) := (-1)^k \chi \left(H_* \left(\frac{(\mathcal{N}_{-k} C_*)^G}{(\mathcal{N}_{-k-1} C_*)^G} \right) \right) + \sum_{q \geq k+1} \beta_q(X^G).$$

By the additivity of the virtual Betti numbers and the exactness of the long homology sequence of 4.3, these new invariants are additive on the varieties where they are defined.

We show below that the $'B_k^G$'s coincide with G. Fichou's equivariant virtual Betti numbers ([5]) in some cases. Namely, in these cases, we verify that $'B_k^G(X) = \dim_{\mathbb{Z}_2} H_k(X; G)$ for any compact nonsingular real algebraic G -variety X .

Corollary 4.12. *Assume $k < 0$. Then for every G -real algebraic variety X , $'B_k^G(X)$ is well-defined and*

$$'B_k^G(X) = \sum_q \beta_q(X^G) = \beta_k^G(X).$$

Proof. Let X be a G -variety. We have

$$'B_k^G(X) = \sum_{q \geq k+1} \beta_q(X^G) = \sum_{q \geq 0} \beta_q(X^G).$$

Now assume that X is compact nonsingular. Then the fixed points set X^G is also compact nonsingular and

$$'B_k^G(X) = \sum_q \beta_q(X^G) = \sum_q \dim_{\mathbb{Z}_2} H_q(X^G) = \dim_{\mathbb{Z}_2} H_k(X; G),$$

since $k < 0$, by lemma 3.14. □

When X is a compact real algebraic variety equipped with a fixed-point free involution, the $'B_k^G(X)$'s and the $\beta_k^G(X)$'s coincide in all degrees, since they are both equal to the virtual Betti numbers of the arc-symmetric quotient of X by the action of G :

Corollary 4.13. *Let X be a compact real algebraic variety equipped with a fixed-point free action of $G = \mathbb{Z}/2\mathbb{Z}$. Then, the $'B_k^G(X)$'s are well-defined for all $k \in \mathbb{Z}$, and we have*

$$'B_k^G(X) = \beta_k(X/G) = \beta_k^G(X).$$

Proof. Fix $k \in \mathbb{Z}$. By proposition 2.7 ([13] Proposition 5.6), there is an isomorphism of complexes

$$(\mathcal{N}_{-k}C_*(X))^G \cong \mathcal{N}_{-k}C_*(X/G).$$

Then all homology groups $H_n \left(\frac{(\mathcal{N}_{-k}C_*)^G}{(\mathcal{N}_{-k-1}C_*)^G} \right) = H_n \left(\frac{\mathcal{N}_{-k}C_*(X/G)}{\mathcal{N}_{-k-1}C_*(X/G)} \right)$ are finite-dimensional and, because $X^G = \emptyset$, we have

$$\begin{aligned} {}'B_k^G(X) &= (-1)^k \chi \left(H_* \left(\frac{\mathcal{N}_{-k}C_*(X/G)}{\mathcal{N}_{-k-1}C_*(X/G)} \right) \right) \\ &= (-1)^k \sum_{\beta} (-1)^{\beta} \dim_{\mathbb{Z}_2} \tilde{E}_{-k+\beta,k}^2(X/G) \\ &= \beta_k(X/G). \end{aligned}$$

Finally, $\beta_k(X/G) = \beta_k^G(X)$ by [5] Proposition 3.15. □

We summarize these results in the following statement

Theorem 4.14. *For $G = \mathbb{Z}/2\mathbb{Z}$, we have*

$${}'B_k^G(X) = \beta_k^G(X)$$

if

- $k < 0$ and X is any real algebraic G -variety,
- X is a compact real algebraic G -variety equipped with a fixed-point free action, for all $k \in \mathbb{Z}$,
- $\dim X = d$ and $k = d$,
- $\dim X = 1$ for all $k \in \mathbb{Z}$.

Proof. We prove the third and fourth points.

First, let X be a real algebraic G -variety of dimension d . Then

$$H_n \left(\frac{(\mathcal{N}_{-d}C_*(X))^G}{(\mathcal{N}_{-d-1}C_*(X))^G} \right) = \begin{cases} (\mathcal{N}_{-d}C_d(X))^G & \text{if } n = d, \\ 0 & \text{otherwise,} \end{cases}$$

and

$${}'B_d^G(X) = \dim_{\mathbb{Z}_2} (\mathcal{N}_{-d}C_d(X))^G$$

(the space $\mathcal{N}_{-d}C_d(X)$ is included in $\ker \partial_d = H_d(X)$ which is finite-dimensional, and the fixed-points set X^d is of dimension $\leq d$).

Assume X to be compact nonsingular. Then $(\mathcal{N}_{-d}C_d(X))^G = (\ker \partial_d)^G = H_d((C_*(X))^G) = H_d(X; G)$ (3.14).

Now consider X to be a 1-dimensional G -real algebraic variety. Then

$$H_n \left(\frac{(C_*(X))^G}{(\mathcal{N}_{-1}C_*(X))^G} \right) = \begin{cases} \frac{(\ker \partial_1)^G}{(\mathcal{N}_{-1}C_*(X))^G} & \text{if } n = 1, \\ H_0((C_*(X))^G) & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $'B_0^G(X)$ is well-defined. Furthermore, if X is compact nonsingular, we have $H_1 \left(\frac{(C_*(X))^G}{(\mathcal{N}_{-1}C_*(X))^G} \right) = 0$ and

$$'B_0^G(X) = \dim_{\mathbb{Z}_2} H_0((C_*(X))^G) + \dim_{\mathbb{Z}_2} H_1(X^G) = \dim_{\mathbb{Z}_2} H_0(X; G)$$

(in this case, X^G is also compact nonsingular). \square

These results may be a hint to think that these two groups of additive invariants of real algebraic G -varieties might constitute the same one. If this is the case, we still have to show that the invariants B_k^G coincide with equivariant Betti numbers on compact nonsingular varieties. If not, these new additive invariants will enrich our toolbox to study the equivariant geometry of real algebraic varieties with action.

In any case, we can see from this study that future work will have to be done to understand the homology of the complexes $(\mathcal{N}_\alpha C_*)^G$ which seem to be central. Geometric consideration and precise technique seem to be necessary to know in particular if the (equivariant) filtered quasi-isomorphism $\mathcal{N}C_*(X) \rightarrow F^{can}C_*(X)$, for X compact nonsingular, is preserved by the functor Γ^G .

In the following point, we give some way to recover the equivariant virtual Betti numbers for $G = \mathbb{Z}/2\mathbb{Z}$, through the existence of what we call the invariant weight complex : from the induced spectral sequence, we can extract additive invariants, from which we recover the equivariant virtual Betti numbers if we add the virtual Betti numbers of the fixed-points set.

This construction would be inextricably linked to the equivariant weight complex if the invariant weight complex could be realized by the Nash-constructible filtration. This would be the case if we had a filtered quasi-isomorphism $(\mathcal{N}C_*)^G \rightarrow (F^{can}C_*)^G$ on compact nonsingular varieties.

4.4 Invariant weight complex and equivariant virtual Betti numbers

Let $G = \mathbb{Z}/2\mathbb{Z}$.

Using again the equivariant version of the extension criterion of Guillén-Navarro Aznar (2.4), we show the existence of an acyclic and additive functor which extends to all real algebraic G -varieties the functor that associates to a projective nonsingular G -variety its invariant chains under the induced action, equipped with the canonical filtration. This will be possible thanks to the short exact sequences

$$0 \rightarrow H_k((C_*(\tilde{Y}))^G) \rightarrow H_k((C_*(Y))^G) \oplus H_k((C_*(\tilde{X}))^G) \rightarrow H_k((C_*(X))^G) \rightarrow 0$$

for any elementary acyclic square (2.1) in $\mathbf{Sch}_c^G(\mathbb{R})$ (4.15).

Directly from the spectral sequence induced by this invariant weight complex, we extract additive invariants on the category of G -real algebraic varieties (4.18), from which we then recover equivariant virtual Betti numbers for $G = \mathbb{Z}/2\mathbb{Z}$ (4.19).

Proposition 4.15. *Let $\pi : \tilde{X} \rightarrow X$ be the equivariant blow-up of a compact nonsingular G -real algebraic variety X along a nonsingular center Y , globally stable under the action of G . Denote by \tilde{Y} the exceptional divisor. Then, for all $k \in \mathbb{Z}$, there is a short exact sequence*

$$0 \rightarrow H_k((C_*(\tilde{Y}))^G) \rightarrow H_k((C_*(Y))^G) \oplus H_k((C_*(\tilde{X}))^G) \rightarrow H_k((C_*(X))^G) \rightarrow 0.$$

Proof. Fix $k \in \mathbb{Z}$. According to 3.14, we have, for every G -variety Z ,

$$H_k((C_*(Z))^G) = H_k(Z; G) / \bigoplus_{i \geq k+1} H_i(Z^G).$$

But, by [5] Lemma 3.6, the short sequence

$$0 \rightarrow H_k(\tilde{Y}; G) \rightarrow H_k(Y; G) \oplus H_k(\tilde{X}; G) \rightarrow H_k(X; G) \rightarrow 0$$

is exact, and, by [11] Proof of Proposition 2.1, the short sequences

$$0 \rightarrow H_i(\tilde{Y}^G) \rightarrow H_i(Y^G) \oplus H_i(\tilde{X}^G) \rightarrow H_i(X^G) \rightarrow 0,$$

for all $i \geq k + 1$, as well. □

This allows us to show that the functor $(F^{can}C_*)^G$ verifies the condition (F2) of theorem 2.4. Indeed, if we look at McCrory-Parusiński's proof of the existence of the weight complex (proof of Theorem 1.1 of [12]), we see that the check of the condition (F2) for $(F^{can}C_*)^G$ goes the same way as its check for $F^{can}C_*$, and that the true key ingredient is the exactness of the short sequences of 4.15.

Proposition 4.16. *The functor*

$$(F^{can}C_*)^G : \mathbf{V}^G(\mathbb{R}) \longrightarrow H \circ \mathcal{C}; X \mapsto (F^{can}C_*(X))^G (= F^{can}(C_*(X))^G)$$

admits an extension to a functor

$${}_G\mathcal{WC}_* : \mathbf{Sch}_c^G(\mathbb{R}) \longrightarrow H \circ \mathcal{C}$$

defined for all real algebraic G -varieties and all equivariant proper morphisms, verifying the following properties :

1. *Acyclicity : For any acyclic square (2.1) in $\mathbf{Sch}_c^G(\mathbb{R})$, the simple filtered complex of the \square_1^+ -diagram in \mathcal{C}*

$$\begin{array}{ccc} {}_G\mathcal{WC}_*(\tilde{Y}) & \rightarrow & {}_G\mathcal{WC}_*(\tilde{X}) \\ \downarrow & & \downarrow \\ {}_G\mathcal{WC}_*(Y) & \rightarrow & {}_G\mathcal{WC}_*(X) \end{array}$$

is acyclic.

2. *Additivity* : For an equivariant closed inclusion $Y \subset X$, the simple filtered complex of the \square_0^+ -diagram in \mathcal{C}

$${}_G\mathcal{WC}_*(Y) \rightarrow {}_G\mathcal{WC}_*(X)$$

is isomorphic to ${}_G\mathcal{WC}_*(X \setminus Y)$.

Such a functor ${}_G\mathcal{WC}_*$ is unique up to a unique isomorphism of $H \circ \mathcal{C}$.

Proof. The functor $(F^{can}C_*)^G$ factorizes through \mathcal{C} so it is Φ -rectified, and if X and Y are two G -real algebraic varieties, we have $(F^{can}C_*(X \sqcup Y))^G = (F^{can}C_*(X))^G \oplus (F^{can}C_*(Y))^G$ so it verifies the condition (F1) of 2.4.

Furthermore, if we apply this functor to an elementary acyclic square, we obtain the same spectral sequence induced by the associated simple filtered complex as in [12], proof of Theorem 1.1, provided we replace the homology of the chains by the homology of the invariant chains. Consequently, the functor $(F^{can}C_*)^G$ verifies the condition (F2) of 2.4 because the short sequences

$$0 \rightarrow H_k((C_*(\tilde{Y}))^G) \rightarrow H_k((C_*(Y))^G) \oplus H_k((C_*(\tilde{X}))^G) \rightarrow H_k((C_*(X))^G) \rightarrow 0$$

are exact.

Thus, we can apply the extension criterion 2.4 to obtain the result. \square

If X is a real algebraic G -variety, we call ${}_G\mathcal{WC}_*(X)$ the invariant weight complex of X . Its homology is the homology of the invariant chains of X :

Proposition 4.17. *For every real algebraic G -variety X , we have*

$$H_*({}_G\mathcal{WC}_*(X)) = H_*((C_*(X))^G).$$

Proof. The short exact sequences

$$0 \rightarrow C_k(\tilde{X}) \rightarrow C_k(Y) \oplus C_k(\tilde{X}) \rightarrow C_k(X) \rightarrow 0$$

and

$$0 \rightarrow C_k(Y) \rightarrow C_k(X) \rightarrow C_k(X \setminus Y) \rightarrow 0,$$

associated respectively to an acyclic square (2.1) in $\mathbf{Sch}_c^G(\mathbb{R})$ and an equivariant closed inclusion $Y \subset X$, are split through equivariant morphisms (respectively $c \mapsto \pi^{-1}c$ and $c \mapsto \bar{c}$).

As a consequence, the sequences of complexes

$$0 \rightarrow (C_*(\tilde{X}))^G \rightarrow (C_*(Y))^G \oplus (C_*(\tilde{X}))^G \rightarrow (C_*(X))^G \rightarrow 0$$

and

$$0 \rightarrow (C_*(Y))^G \rightarrow (C_*(X))^G \rightarrow (C_*(X \setminus Y))^G \rightarrow 0$$

are exact, that is the functor $(C_*)^G : \mathbf{Sch}_c^G(\mathbb{R}) \rightarrow H \circ \mathcal{D}$; $X \mapsto (C_*(X))^G$ verifies the acyclicity and additivity conditions. This also the case of $\varphi \circ {}_G\mathcal{WC}_* : \mathbf{Sch}_c^G(\mathbb{R}) \rightarrow H \circ \mathcal{D}$ (where $\varphi : H \circ \mathcal{C} \rightarrow H \circ \mathcal{D}$ is the functor induced by the forgetful functor of the filtration $\mathcal{C} \rightarrow \mathcal{D}$). Finally, those two functors are both extensions of $(C_*)^G : \mathbf{V}^G(\mathbb{R}) \rightarrow H \circ \mathcal{D}$. Thus, by the unicity brought by the equivariant extension criterion 2.4, they are quasi-isomorphic. \square

We read the acyclicity and additivity conditions of the filtered invariant weight complex on the induced spectral sequence, called the invariant spectral sequence and denoted by ${}_G E$, which converges to the homology of invariant chains. Having the same boundings as the weight spectral sequence, we can then extract from this invariant spectral sequence some additive invariants which coincide with the dimension of the homology of the invariant chains on compact nonsingular varieties :

Proposition 4.18. *We denote by ${}_G \tilde{E}$ the reindexed invariant spectral sequence obtained by putting $p' = 2p + q$, $q' = -p$ et $r' = r + 1$. For all real algebraic G -varieties X and all $q \in \mathbb{Z}$, we define*

$${}_G \beta_q(X) := \sum_p (-1)^p \dim {}_G \tilde{E}_{p,q}^2(X).$$

The invariants ${}_G \beta_q$ are additive on the category of G -real algebraic varieties and, if X is compact nonsingular, ${}_G \beta_q(X) = \dim_{\mathbb{Z}_2} H_q((C_(X))^G)$.*

Proof. First of all, the fact that, if X is a G -variety of dimension d , the terms ${}_G \tilde{E}_{p,q}^2(X)$ are all finite-dimensional and bounded in a triangle with vertices $(0,0)$, $(0,d)$ and $(d,0)$ comes from the isomorphism between the invariant weight spectral sequence and a spectral sequence associated to an equivariant cubical hyperresolution, analog to the one in the non-equivariant frame of McCrory and Parusiński (see [12] Proposition 1.8. and Corollary 1.10.), considering this time invariant chains.

In the same way as in [12] section 1.3, the additivity of the ${}_G \beta_q$'s is deduced from the additivity condition of the invariant weight complex, and the second condition comes from the fact that whenever X is compact nonsingular, the invariant weight complex of X is quasi-isomorphic in \mathcal{C} to $(F^{can} C_*(X))^G$. To show this, use the extension property of the inclusion of categories $\mathbf{V}^G(\mathbb{R}) \rightarrow \mathbf{Reg}_{comp}^G(\mathbb{R})$, and the additivity and acyclicity of the functor $(F^{can} C_*)^G$ in $\mathbf{Reg}_{comp}^G(\mathbb{R})$. \square

Finally, we can use these additive invariants ${}_G \beta_q$ to recover Fichou's equivariant virtual Betti numbers :

Theorem 4.19. *For every real algebraic G -variety X , we have*

$$\beta_q^G(X) = {}_G \beta_q(X) + \sum_{i \geq q+1} \beta_i(X^G)$$

for all $q \in \mathbb{Z}$.

Proof. Fix $q \in \mathbb{Z}$. The additivity of the right member comes from the additivity of the invariants ${}_G \beta_q$ and the additivity of the virtual Betti numbers.

Then, if X is compact nonsingular,

$${}_G \beta_q(X) + \sum_{i \geq q+1} \beta_i(X^G) = \dim_{\mathbb{Z}_2} H_q((C_*(X))^G) + \sum_{i \geq q+1} \dim_{\mathbb{Z}_2} H_i(X^G) = \dim_{\mathbb{Z}_2} H_q(X; G)$$

(according to lemma 3.14). \square

Remark 4.20. Note that for all G -varieties X and all k in \mathbb{Z} ,

$${}_G\beta_k(X) = (-1)^k \chi \left(H_* \left(\frac{{}_G\mathcal{W}_{-k}C_*(X)}{{}_G\mathcal{N}_{-k-1}C_*(X)} \right) \right).$$

Consequently, if the Nash-constructible filtration was to realize the invariant weight complex, it would be the link between the equivariant and invariant weight complexes, spectral sequences, filtrations. Furthermore, we would have

$$'B_k^G(X) = {}_G\beta_k(X) + \sum_{i \geq q+1} \beta_i(X^G) = \beta_q^G(X)$$

for $G = \mathbb{Z}/2\mathbb{Z}$

From this last result 4.19 and from the previous ones of this subsection 4.4, it should be again stressed that it seems very important for the study of real algebraic G -varieties (even with G any finite group) to consider both equivariant homology and homology of the invariant chains, keeping in mind the strong link between them.

References

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Fabien PRIZIAC
 Université de Bretagne Occidentale
 Laboratoire de Mathématiques de Bretagne Atlantique
 6, avenue Le Gorgeu CS 93837
 29238 BREST Cedex 3 (France)
 fabien.priziac@univ-brest.fr